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A Supply Chain Equilibrium Model with General Price-Dependent Demand

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ABSTRACT

The concept of supply chain equilibrium has been widely employed to solve real-life cases. Under this concept, decision makers move simultaneously and compete in a noncooperative manner to achieve a supply chain network equilibrium. This paper proposes a supply chain network equilibrium model consisting of multiple raw material suppliers, manufacturers and retailers. Unlike previous studies, we assume that the demand for the product at each retail outlet is modeled as general stochastic functions of price that encompass additive-multiplicative demand models used in previous studies. Under general price-dependent demand functions, we derive the optimality conditions of suppliers, manufacturers and retailers, and establish that the governing equilibrium conditions can be formulated as a finite-dimensional variational inequality problem. The existence and uniqueness of the solution to the variational inequality are examined. A sensitivity analysis and a series of numerical tests are conducted to illustrate the analytical effects of demand distribution, model parameters, demand level and variability on quantity shipments, prices, and expected profits. Managerial insights are reported to show the impact of different types of demand functions and model parameters on the equilibrium solutions.

1. Introduction

The equilibrium concept in supply chains is drawn from network economics [1] and assumes a simultaneous move of the various decision-makers to achieve a supply chain network equilibrium. In the field of supply chain management, this concept is practically relevant and has been adopted to solve real-life cases. [2] develop a food supply chain equilibrium model for fresh product items, such as vegetables and fruits. The model was used to analyse various scenarios prior/during/after a foodborne disease outbreak within the cantaloupe market in the United States. [3] propose a multитiered supply chain network equilibrium model for disaster relief with capacitated freight service provision. The model was applied to a case study on an international health crisis in western Africa to examine the impacts of adding a freight service provider and an humanitarian organization on the profits of freight service providers and on the costs incurred by the humanitarian organizations. Other relevant applications of the supply chain equilibrium concept includes electronic waste recycling [4], internet advertising [5], pharmaceutical products [6], green technology investment [7], and agricultural products [8].

Most of the studies dealing with supply chain equilibrium do not consider the effect of demand uncertainty on the equilibrium solutions. However, possessing relevant demand information such us density functions can assist operations supply chain managers to jointly compute optimal order quantities and prices before demand is realized. In this paper, we develop a new supply chain equilibrium model under demand uncertainty in which the demand for the product at each retailer is modeled by a general demand distribution and depends on all retailer prices and a on random variable independent of the price with increasing failure rate (IFR). This general demand model encompasses all common demand functions adopted in the literature [9,10] including additive linear, multiplicative isoelastic, power, logit, exponential, logarithmic, and mixed additive-multiplicative functions. The results of the supply chain equilibrium model are used to address the following research questions:

1. How do model parameters affect equilibrium solutions and expected profits of the supply chain members?
2. How the optimal quantity shipments, prices, and expected profits are influenced by the choice of the demand model?
3. What is the effect of demand variability on each supply chain member’s expected profit? Is this demand variability effect the same across different demand models?

Similar to [11], we adopt the concept of LSR elasticity and make realistic assumptions on the demand functions. Under these model assumptions, we demonstrate the pseudo-concavity of the retailers’ expected profits as functions of prices. This allows us to characterize the equilibrium conditions of all raw material suppliers, manufacturers and retailers as a variational inequality in which they should determine their own optimal decision variables, given the optimal ones of competitors. As far as we are aware, this is the first supply chain equilibrium model to consider general price-dependent demand including all common demand functions adopted in the extant literature.

The main contribution of this paper is threefold. First, we develop a new supply chain equilibrium model that incorporates extended price-dependent stochastic demand functions and price competition among retailers. Second, we propose a new variational inequality formulation in which raw material suppliers, manufacturers and retailers should determine their optimal decision variables, given the optimal ones of competitors, and demonstrate the existence and uniqueness of the solution to this variational inequality. Third, through numerical tests and sensitivity analysis, we illustrate the analytic effects and practical managerial implications of different types of demand functions, model parameters, demand level, and demand variability on quantity shipments, prices, and expected profits.

The rest of this paper is organized as follows. Section 2 positions our research with respect to other contributions in the literature. Section 3 presents the optimality conditions of the raw material suppliers, manufacturers and retailers using the variational inequality theory. Section 4 provides the equilibrium conditions of the supply chain network model and its qualitative properties. Section 5 discusses examples of general demand models. Section 6 examines structural properties of the equilibrium solutions. Section 7 provides numerical examples to illustrate the effects of demand distribution, model parameters, demand level, and demand variability on quantity shipments, prices, and expected profits. Section 8 discusses some important managerial insights. Finally, Section 9 provides concluding remarks and ideas for future research. All proofs are provided in the appendices.

2. Positioning in literature

In determining multi-echelon supply chain optimal decisions, the game-theory-based framework used to model supply chain depends on the power relation between its members, see [12,13]. The first approach, assumes that decisions makers have similar powers, move simultaneously and compete in a noncooperative manner to achieve a supply chain network equilibrium. See [14] for review on this topic. Real-life applications of this approach in different fields were discussed in the Introduction section. Most studies have adopted a variational inequality (VI) formulation to characterize the equilibrium solutions of the various decision-makers. [15] were the first to develop an equilibrium model of a competitive supply chain network involving multiple manufacturers, retailers, and consumers in demand markets. This model provides the foundation of supply chain equilibrium models and has been extended by several authors to include capacity constraints [16, 17], closed-loop supply chains [18,19], and multi-period supply chain networks [17,19,20].

The second approach, assumes that one member of the supply chain is more powerful and acts as a leader in the decision making process. A Stackelberg game theoretical framework with leaders and followers is used to find the optimal decisions. A thorough review of this approach can be found in [21,22]. Applications of this approach are found in various fields. [23] gave an application for a food supply chain dealing with pork meat industry. [24] uses this approach in a supply chain where it is desired to maximize profit and corporate social responsibility (CSR). In [25], the authors utilize social work donation and recycling investment as tools of corporate social responsibility and integrate the CSR investments into the optimization of a closed loop supply chain. [26] adopts this approach in a hospital supply chain.

Most literatures, dealing with Stackelberg Equilibrium in supply chains, notice that optimizing individual member objectives does not necessarily lead the optimality for the whole supply chain and that a centralized decision making leads to larger total supply chain profit. To solve such issue, many different type of contracts/incentives were introduced and aimed at inducing supply chain coordination. Literature dealing with supply chain coordination is quite extensive, see [27–29] for reviews. To cite some applications, one see that [30] uses price, rebate and returns supply contracts to coordinate supply chains while [31] adopts a hybrid all-unit quantity discount along a franchise fee contract for supply chain coordination. [32] examines the effect of channel leadership and information asymmetry on supply chain coordination. [33] uses product recycling and explores channel coordination in a socially responsible manufacturer-retailer closed-loop supply chain. The paper [34] proposes two hybrid contract bargaining processes that can be used for channel coordination of a supply chain that deals with a deteriorating product.

Our manuscript fits in the first approach and assumes that supply chain members move simultaneously in order to reach an equilibrium. Previous studies in this area utilize a projection-based algorithm to compute equilibrium shipments and prices but do not consider the effect of demand uncertainty on the equilibrium solutions. In fact, although the demand for a product may not be known with certainty but we may possess some information such as the density functions based on historical/forecasted data that allows decision makers to jointly determine order quantities and price before demand is realized. This is known in operations research literature as the newsvendor problem with pricing decisions [11,35,36].

The extant literature on the newsvendor problem with pricing (NVP) is extensive but mainly focuses on the additive and multiplicative models. In the additive and multiplicative demand cases, demand is represented as the sum and the product, respectively, of a deterministic price-dependent demand function and a random term that is independent of price. For the additive demand model, price affects the location of the demand distribution but not the demand variance while for the multiplicative case, price affects the scale of the distribution but not the coefficient of variation. [35] provide a comprehensive review of these two types of models when the mean demand is linear in the additive case and exponential (iso-elastic) in the multiplicative case. [36] and [37] study the NVP problem with additive and multiplicative demand models when the mean demand has increasing price elasticity (IPE) and the random noise has generalized increasing failure rate (GIFR). The authors prove that under these conditions, the expected profit of the newsvendor is unimodal or quasi-concave with respect to the price. In contrast to previous studies, [11] consider general price-dependent demand functions that include additive-multiplicative demand models as well as other relevant demand models such as logit, exponential, and power functions. Using a new measure, called the lost-sale rate (LSR) elasticity, they provide necessary and sufficient conditions for the NVP optimal policy under both coordinated and sequential decisions. Throughout the paper, the authors assume that riskless unconstrained revenue is strictly concave with respect to price. However, this assumption is not satisfied for logit, power, and iso-elastic demand functions which is a significant drawback of the paper. [38] consider a periodic review of the NVP problem under general price-dependent demands. For both the backorder and lost sales models, the objective is to maximize the expected profit over a finite selling horizon by coordinating the inventory and pricing decisions in each period. The authors utilize a new concept of upper-set and lower-set decreasing properties (USDP/LSDP) to derive sufficient conditions for the optimality of a base-stock list price policy based on the strict monotonicity of demand functions. Although the USDP/LSDP properties are considered to be small relaxations of the
strict concavity of the riskless revenue, some demand functions such as the logit and exponential functions do not share these properties, limiting the applicability of the model.

As to supply chain equilibrium models under demand uncertainty, [39] develop a two-echelon model with multiple manufacturers and retailers. The retailers are faced with random demand and seek to maximize their expected profits with a penalty associated with a shortage and excess supply. The authors formulate the optimality conditions of the retailers as a variational inequality when the retailers first decide on the optimal amount to order from manufacturers. The equilibrium demand prices are then derived by assuming that the total quantity purchased by each retailer from all manufacturers is equal to the expected demand at that retail outlet. This constitutes a main limitation because at equilibrium, the total actual demand is not necessarily equal to the total supply. The model of [39] was extended by [40] for multi-commodity flow and by [41] within closed-loop supply chains. [42] propose a dynamic supply chain equilibrium model for a closed-loop supply chain under uncertain and time-dependent demands and returns. The seasonality of demand is modeled by assuming that the expected demand at each retail outlet is not influenced by the retailer’s own selling price, but also by the price set by competitors. The demand for the product at each retail outlet depends on all retailer prices and the random noise has increasing failure rate/generalized increasing failure rate (IFR/GIFR).

3. The supply chain network model with general price-dependent demand

As shown in Figure 1, we consider a supply chain network consisting of N raw material suppliers who are involved in supplying one raw material to I manufacturers. The manufacturers produce a homogeneous product that can then be purchased by J retailers. We assume a one-to-one ratio between the raw material and product and this assumption can be easily relaxed by considering a non one-to-one ratio. Each retail outlet makes the product available to consumers in its own demand market. The links in the supply chain network denote transportation/transaction links. As assumed in the supply chain equilibrium literature, manufacturers must agree with the raw material suppliers as to the volume of shipments and retailers must agree with the manufacturers as to the purchasing prices which shall be determined using equilibrium conditions. In addition, all the supply chain members move simultaneously and compete in a noncooperative manner under the Cournot-Nash equilibrium framework, meaning decision makers will determine their own optimal decision variables, given the optimal ones of the competitors. The demand for the product at each retail outlet is modeled using a general demand distribution.

All indices, parameters and variables in the supply chain equilibrium network are defined as follows.

Indices

\( n \): index of raw material suppliers in the SC network, \( n = \{ 1, \ldots, N \} \).
\( i \): index of manufacturers in the SC network, \( i = \{ 1, \ldots, I \} \).
\( j \): index of retailers in the SC network, \( j = \{ 1, \ldots, J \} \).

Parameters

\( c_i \): per-unit handling cost at retailer \( j \).
\( \lambda_i^j \): per-unit salvage value of having excess supply at retailer \( j \).
\( \lambda_j^i \): per-unit shortage cost of having excess demand at retailer \( j \).

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![Fig. 1. The supply chain network](image-url)
Variables

$q_i^c$: nonnegative raw material shipment from supplier $n$ to manufacturer $i$. Group the shipments of all the suppliers into the column vector $q_1 \in R^{n_i}$.

$q_i^p$: nonnegative product shipment from manufacturer $i$ to retailer $j$. Group the shipments of all the manufacturers into the column vector $q_2 \in R^{j}$.

$p_n^e$: selling price from supplier $n$ to manufacturer $i$.

$p_j$: purchasing price at retailer outlet $j$. Group the prices of all the retailers into the column vector $p \in R^{j}$.

$p_j$: selling price at retailer outlet $j$. Group the prices of all the retailers into the column vector $p \in R^{j}$.

In the following subsections, we derive the optimality conditions of the raw material suppliers, manufacturers and retailers.

3.1. Raw Material Suppliers and their equilibrium conditions

Each raw material supplier $n$ decides on the amount of raw material $\tilde{q}_n$ to ship to each manufacturer $i$. Raw material supplier $n$ incurs a procurement and a transaction cost, $c_n(q_n)$, with each manufacturer $i$. Given the above cost, we can express the criterion of profit maximization for each raw material supplier $n$ as:

$$
\max_{\tilde{q}_n} \Pi_n = \sum_{i=1}^{N_i} \tilde{p}_n^i \tilde{q}_n - \sum_{i=1}^{N_i} c_n(q_n) \\
\text{subject to: } \tilde{q}_n \geq 0. \tag{1}
$$

Equation (1) states that supplier $n$’s profit equals sales revenue less costs associated with procurement and transaction. Note that $\tilde{p}_n^i$ denote the optimal prices from each raw material supplier $n$ to each manufacturer $i$. We will show later how to recover these optimal prices after solving the complete supply chain equilibrium model.

We assume that procurement and transaction cost functions for each raw material supplier are continuous and convex. Therefore, the optimality conditions for all raw material suppliers can be expressed simultaneously as the following variational inequality [43]: Determine $q_1^* \in R^{n_i}$ satisfying:

$$
\sum_{n=1}^{N_i} \sum_{i=1}^{N_i} \left[ \frac{\partial c_n(q_n)}{\partial q_n} \right] (\tilde{q}_n) \times (\tilde{q}_n - q_n^*) \geq 0 \quad \forall q_1 \in R^{n_i}. \tag{2}
$$

3.2. Manufacturers and their equilibrium conditions

Each manufacturer $i$ must decide on the amount of raw material $\tilde{q}_n$ to get from each supplier $n$ and the amount of product $q_2$ to ship to each retailer $j$. Raw material suppliers and manufacturers should agree on the quantities $\tilde{q}_n$ and manufactures and retailers should also agree on the prices $p_j$ which shall be determined using equilibrium conditions. The model assumes that retailer $j$ pays the same price $p_j$ to all manufacturers. This assumption is realistic since the model does not consider capacity constraints and any manufacturer $i$ setting a higher price $p_j$ than the equilibrium price $p_j^*$ would induce retailer $j$ to not purchase any quantity from that manufacturer. Manufacturer $i$ incurs a production and a transaction cost, $c_i(q_i^c)$, with each retailer $j$. Given the above cost, we can write the objective of each manufacturer as:

$$
\max_{q_i^c, q_j} \Pi_i = \sum_{j=1}^{J_i} p_j q_j - \sum_{j=1}^{J_i} c_i(q_i^c) - \sum_{n=1}^{N_i} \tilde{p}_n^i \tilde{q}_n. \tag{3}
$$

Equation (3) states that manufacturer $i$’s profit equals sales revenue less costs associated with production and transaction, and payout to raw material suppliers. Constraint (4) states that the sum of all shipment quantities to retailers must be less or equal to the quantities procured from the raw material suppliers.

We assume that production and transaction cost functions for each manufacturer are continuous and convex. Therefore, the optimality conditions for all manufacturers can be expressed simultaneously as the following variational inequality: Determine $(q_1^*, q_2^*) \in \Lambda \cap R_{+}^{n_i+j}$ satisfying:

$$
\sum_{n=1}^{N_i} \sum_{j=1}^{J_i} \tilde{p}_n^i \times \left[ \tilde{q}_n - q_n^* \right] + \sum_{j=1}^{J_i} \sum_{i=1}^{N_i} \left[ \frac{\partial c_n(q_i^c)}{\partial q_i^c} \right] (q_i^c) \times (q_i^c - q_i^*) \\
\text{subject to: } \sum_{j=1}^{J_i} q_j \leq \sum_{n=1}^{N_i} \tilde{q}_n. \tag{4}
$$

for all $(q_1, q_2) \in \Lambda$ where $\Lambda$ is the convex set given by:

$$\Lambda = \left\{ (q_1, q_2) \in R_{+}^{n_i+j} : \sum_{j=1}^{J_i} q_j - \sum_{n=1}^{N_i} \tilde{q}_n \leq 0, \forall 1 \leq i \leq I \right\} \tag{5}
$$

3.3. Retailers and their equilibrium conditions

Each retailer $j$ has to decide on the total amount to purchase from manufacturers and the selling price to consumers while simultaneously seeking to reach a Nash equilibrium under demand uncertainty. The demand for the product at each retailer $j$, $D_j(p_j, \epsilon_j)$, is assumed to follow a general demand distribution that depends on the whole vector price $p$ and on a random variable $\epsilon_j$, independent of $p$, defined on the range $[A_j, B_j]$. It can be seen that the classical additive demand model ($D_j(p_j, \epsilon_j) = d(p_j) + \epsilon_j$) and multiplicative model ($D_j(p_j, \epsilon_j) = d(p_j \epsilon_j)$) are simply special cases of the general demand model. If $\Lambda = \{ q_1, q_2 \} \in R_{+}^{n_i+j}$ denotes the total supply at retailer $j$ obtained from all the manufacturers, then if demand for the product does not exceed $s_j$, the revenue of retailer $j$ is $p_jD_j(p_j, \epsilon_j)$ and most of the $s_j - D_j(p_j, \epsilon_j)$ leftovers is disposed at the unit salvage value $\lambda^s_j$. Alternatively, if demand exceeds $s_j$, then the revenue of retailer $j$ is $p_jD_j(p_j, \epsilon_j)$ and each of the $D_j(p_j, \epsilon_j) - s_j$ shortages incurs a per-unit shortage cost $\lambda^s_j$. Using the notation $Q_j = (q_1, q_2)$, the profit of retailer $j$, $W_j(Q_j, p_j)$, representing the difference between sales revenue and total costs, is given by:

$$W_j(Q_j, p_j) = \begin{cases} p_jD_j(p_j, \epsilon_j) - c_j s_j - \rho_j s_j - \lambda^s_j [s_j - D_j(p_j, \epsilon_j)] & \text{if } D_j(p_j, \epsilon_j) \leq s_j \\ p_jD_j(p_j, \epsilon_j) - c_j s_j - \rho_j s_j - \lambda^s_j [D_j(p_j, \epsilon_j) - s_j] & \text{if } D_j(p_j, \epsilon_j) > s_j \end{cases} \tag{6}
$$

Assuming $D_j(p, x)$ to be strictly monotone in $x$ and defining $z_j = s_j(p_j, s_j)$ as the unique solution of $D_j(p_j, z_j) = s_j$, the profit $W_j(Q_j, p_j)$ reduces to:

$$W_j(s_j, p_j) = \begin{cases} (p_j + \lambda^s_j - c_j - \rho_j) s_j - (p_j + \lambda^s_j - \lambda_j^s) [s_j - D_j(p_j, \epsilon_j)] & \text{if } D_j(p_j, \epsilon_j) \leq s_j \\ (p_j + \lambda^s_j - \lambda_j^s) D_j(p_j, \epsilon_j) & \text{if } D_j(p_j, \epsilon_j) > s_j \end{cases}$$

Each retailer $j$ seeks to maximize the expected profit $\Pi_j(s_j, p_j) = E[W_j(s_j, p_j)]$. More precisely, retailer $j$ is trying to find $s_j$ and $p_j$ that maximize $\Pi_j(s_j, p_j) = \begin{cases} (p_j + \lambda^s_j - c_j - \rho_j) s_j - (p_j + \lambda^s_j - \lambda_j^s)s_jF_j(z_j) & \text{if } D_j(p_j, \epsilon_j) \leq s_j \\ (p_j + \lambda^s_j - \lambda_j^s) \int_{s_j}^{\hat{s}_j} D_j(p_j, \epsilon_j) \epsilon_j f(\epsilon_j) d\epsilon_j + \lambda_j^s \int_{\hat{s}_j}^{\hat{s}_j} D_j(p_j, \epsilon_j) f(\epsilon_j) d\epsilon_j & \text{if } D_j(p_j, \epsilon_j) > s_j \end{cases}$.

To ensure the existence and uniqueness of an optimal solution, the following assumptions are needed:

**Assumption 1.** For any retailer $j = 1, 2, \ldots, J$, the random variable $\epsilon_j$
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satisfies the following properties:

1. \( \epsilon_j \) has a continuous distribution \( F_j(x) \) with density \( f_j(x) \).
2. The failure rate function of \( \epsilon_j \)'s distribution, \( \eta(x) = \frac{f(x)}{1-F(x)} \), is increasing.

As discussed in [43], classes of increasing failure rate (IFR) distributions include: Uniform, Normal (as well as truncated Normal at zero), Exponential, Gamma (with shape parameter \( s \geq 1 \)), Beta (with parameters \( (r,s) \) both being \( \geq 1 \)), and Weibull distribution (with shape parameter \( s \geq 1 \)).

Let \( \eta_j(p,x) = \frac{\partial \eta_j(p,x)}{\partial p_j} \) denote the price elasticity of \( D_j \). Price elasticity measures the percentage change in demand in response to a percentage change in retailer price \( p_j \). We also define, for each \( k \neq j \), the cross price elasticities \( \eta_{jk}(p,x) = \frac{\partial \eta_j(p,x)}{\partial p_k} \) that measure the percentage change in demand in response to a percentage change in other retailer prices \( p_k \). As adopted by [11], we define \( \ell_j(p,x) = -\frac{\partial \eta_j(p,x)}{\partial p_j} \) as the lost-sales rate (LSR) elasticity that measures the percentage change in the rate of lost sales with respect to the percentage change in price \( p_j \) for a given quantity \( x \).

**Assumption 2.** For any retailer \( j = 1, 2, \ldots, J \), demand \( D_j(p,x) \) satisfies the following properties:

1. \( \frac{dD_j(p,x)}{dx} \geq 0, \frac{d^2D_j(p,x)}{dp_j dx} \leq 0, \frac{d^2D_j(p,x)}{dp^2_j} \leq 0, \forall k \neq j, \sum_{k=1}^{J} \frac{dD_j(p,x)}{dp_k} \leq 0. \)
2. \( \frac{dD_j(p,x)}{dp_j} \leq 0, \frac{d^2D_j(p,x)}{dp^2_j} \leq 0, \forall k \neq j, \frac{d^2D_j(p,x)}{dp^2_j} + \sum_{k=1}^{J} \frac{dD_j(p,x)}{dp_k} \leq 0. \)
3. \( \frac{d^2D_j(p,x)}{dp^2_j} \leq 0, \frac{d^2D_j(p,x)}{dp^2_j} \geq 0 \) and \( \frac{\partial D_j(p,x)}{\partial p_j} + \sum_{k=1}^{J} \frac{\partial D_j(p,x)}{\partial p_k} \leq 0, \frac{\partial D_j(p,x)}{\partial p_j} \) and \( \frac{\partial^2 D_j(p,x)}{\partial p^2_j} \) are increasing in \( x \) and \( \frac{\partial^2 D_j(p,x)}{\partial p^2_j} \) is decreasing in \( x \).
4. \( \frac{d^2D_j(p,x)}{dp^2_j} \geq 0, \frac{d^2D_j(p,x)}{dp^2_j} \leq 0 \) and \( \frac{\partial D_j(p,x)}{\partial p_j} + \sum_{k=1}^{J} \frac{\partial D_j(p,x)}{\partial p_k} \leq 0, \frac{\partial D_j(p,x)}{\partial p_j} \) and \( \frac{\partial^2 D_j(p,x)}{\partial p^2_j} \) are decreasing in \( x \), \( \frac{\partial^2 D_j(p,x)}{\partial p^2_j} \) is increasing in \( x \), \( \frac{\partial^2 D_j(p,x)}{\partial p^2_j} \) and \( \frac{\partial^2 D_j(p,x)}{\partial p^2_j} \) are independent of \( x \).

As seen in Section 5, all demand functions used in extant literature satisfy all conditions of Assumption 2. Moreover, each of these demand functions satisfy at least one of the above conditions of Assumption 3, allowing us to include all different types of demand functions in our supply chain equilibrium model.

To seek the optimal solution, note that taking the first derivative of function \( \Pi_j(p) \) with respect to \( s \) and using the definition of \( \eta_j \) yields

\[
\frac{d\Pi_j}{dp_j} = \left( p_j + \lambda_j^* - c_j - \rho_j \right) - \left( p_j + \lambda_j^* - \lambda_j^* \right) F_j(\epsilon_j).
\]

It can be seen that, when \( \lambda_j^* \leq c_j + \rho_j \leq p_j \), the equation \( d\Pi_j/dp_j = 0 \) admits a solution \( \beta_j \), given by \( \beta_j = D_j(p, \beta_j) \), where \( \beta_j = F_j^{-1}(\epsilon_j) \) with \( \epsilon_j = (p_j + \lambda_j^* - c_j - \rho_j)/(p_j + \lambda_j^* - \lambda_j^*). \) Note that the condition \( \lambda_j^* \leq c_j + \rho_j \) amounts to the fact that the salvage value \( \lambda_j^* \) should be less than or equal to the marginal cost \( c_j + \rho_j \) and the condition \( c_j + \rho_j \leq p_j \) ensures that retailer \( j \) is able to make nonnegative profit. Substituting \( \beta_j \) in (6) reduces the retailer problem to

\[
\max_{p_j \in \mathcal{P}_j} \Pi_j(p_j) = \left( p_j + \lambda_j^* - \lambda_j^* \right) \int_{\beta_j}^{p_j} D_j(p_j, x) f_j(x) dx
\]

\[
-\lambda_j^* \int_{\beta_j}^{p_j} D_j(p_j, x) f_j(x) dx,
\]

where \( \mathcal{P}_j(\rho) = \{ p_j \in \mathbb{R}_+: \lambda_j^* \leq c_j + \rho_j \leq p_j \} \), and \( \mathcal{P}_j \) is the maximum admissible price for retailer \( j \). The next theorem shows that the retailer profit function \( \Pi_j(p_j) \) is pseudo-concave.

**Theorem 1.** If the conditions of Assumptions 1, 2, and 3 are satisfied, then the function \( \Pi_j(p_j) \) is pseudo-concave in \( p_j \).

The proof is given in Appendix A.

Next, using Lemma 1 in [49] and Theorem 1 above, the optimality conditions for all retailers could be expressed simultaneously as the following variational inequality: Determine \( p^* \in \Gamma(\rho) \subset \mathbb{R}_+^J \) satisfying

\[
\sum_{j=1}^{J} \left[ -\left( p_j^* + \lambda_j^* - \lambda_j^* \right) \int_{\beta_j}^{p_j^*} \frac{\partial D_j(p_j, x)}{\partial p_j} f_j(x) dx + \lambda_j^* \int_{\beta_j}^{p_j^*} \frac{\partial D_j(p_j, x)}{\partial p_j} f_j(x) dx \right] - \int_{\beta_j}^{p_j^*} D_j(p_j, x) f_j(x) dx - D_j(p_j^*, \zeta_j) \left( 1 - F_j(\zeta_j) \right) \right) \geq 0, \forall p \in \Gamma(\rho),
\]

where \( \gamma_j = F_j^{-1}(\zeta_j), \) \( \zeta_j = (p_j^* + \lambda_j^* - \lambda_j^*)/p_j^* + \lambda_j^* - \lambda_j^* \) and \( \Gamma(\rho) = \bigotimes_{j=1}^{J} \Gamma_j(\rho) \).

\[
\frac{d^2D_j(p,x)}{dp^2_j} \leq 0, \frac{d^2D_j(p,x)}{dp^2_j} \geq 0 \text{ and } \frac{\partial D_j(p,x)}{\partial p_j} + \sum_{k=1}^{J} \frac{\partial D_j(p,x)}{\partial p_k} \leq 0, \frac{\partial D_j(p,x)}{\partial p_j} \text{ and } \frac{\partial^2 D_j(p,x)}{\partial p^2_j} \text{ are increasing in } x, \frac{\partial^2 D_j(p,x)}{\partial p^2_j} \text{ is decreasing in } x.
\]

\[
\frac{d^2D_j(p,x)}{dp^2_j} \leq 0, \frac{d^2D_j(p,x)}{dp^2_j} \geq 0 \text{ and } \frac{\partial D_j(p,x)}{\partial p_j} + \sum_{k=1}^{J} \frac{\partial D_j(p,x)}{\partial p_k} \leq 0, \frac{\partial D_j(p,x)}{\partial p_j} \text{ and } \frac{\partial^2 D_j(p,x)}{\partial p^2_j} \text{ are increasing in } x, \frac{\partial^2 D_j(p,x)}{\partial p^2_j} \text{ is decreasing in } x.
\]
4. Equilibrium conditions of the supply chain

4.1. Equilibrium conditions

As in the supply chain equilibrium literature, the sum of the optimality conditions for all raw material suppliers, as expressed by inequality (2), the sum of the optimality conditions for all manufacturers, as expressed by inequality (5) and the optimality conditions for all retailers, as expressed by inequality (8) must be satisfied. In addition, the amounts of the raw materials that the suppliers ship to the manufacturers must be equal to the shipments that the manufacturers accept from suppliers. Moreover, the amounts of the product that the manufacturers ship to the retailers must be equal to the total amounts purchased by the retailers, as expressed in the following condition:

\[
s'_j = D_j \left( p' \cdot F_j^{-1} \left( \frac{p'_j - c_j - p'_i + \lambda'_j}{p'_j + \lambda'_j} \right) \right) = \sum_{i=1}^{I} q'_i
\]  

(9)

Condition (9) states that when equilibrium price \( p'_j \) that retailer \( j \) pays for the product is positive, then the supply \( s'_j \) needed for at the retailer outlet is positive and must be equal to the total quantities purchased from all manufacturers. It can then be expressed as the following variational inequality: Determine \((q'_1, q'_2, \ldots, q'_P) \in \mathbb{R}^P \times \mathbb{R}^{I \times \Omega} \) satisfying

\[
\sum_{i=1}^{I} \sum_{j=1}^{P} q_{ij} - D_i \left( p' \cdot F_i^{-1} \left( \frac{p'_i - c_i - p'_j + \lambda'_i}{p'_i + \lambda'_i} \right) \right) \times \left[ p_i - p'_j \right] \geq 0
\]  

(10)

\( \forall (q_1, q_2, p, \rho) \in \mathbb{R}^P \times \mathbb{R}^{I \times \Omega} \) and \( \Gamma = \bigotimes_{j=1}^{I} \Gamma_j \) and \( \Gamma_j = \{ (p_j, \rho) \in \mathbb{R}^2 | \lambda_j \leq c_j + \rho_j \leq p_j \leq \bar{p}_j \} \).

The summation of inequalities (2), (5), (8), and (10) yields the following theorem:

**Theorem 2.** The equilibrium conditions governing the supply chain model with general price-dependent demand are equivalent to the solution of the variational inequality problem given by: Determine \((q'_1, q'_2, \ldots, q'_P) \in \mathbb{R}^P \times \mathbb{R}^{I \times \Omega} \) satisfying

\[
\sum_{i=1}^{I} \sum_{j=1}^{P} \frac{\partial c_{ij}}{\partial q_{ij}} (q'_i) \times [q_{ij} - q'_i] + \sum_{j=1}^{P} \sum_{i=1}^{I} \left[ \frac{\partial c_{ij}}{\partial q_{ij}} (q'_i) \times [q_{ij} - q'_i] \right] \times [q_{ij} - q'_i] \]

\[+ \sum_{j=1}^{P} \int_{\lambda_j}^{p_j} D_j \left( p' \cdot F_j^{-1} \left( \frac{p'_j - c_j - p'_i + \lambda'_j}{p'_j + \lambda'_j} \right) \right) \times \left[ p_i - p'_j \right] \geq 0
\]  

(11)

\( \forall (q_1, q_2, p, \rho) \in \Omega = \mathbb{R}^P \times \mathbb{R}^{I \times \Omega}, \text{ where } \Gamma_j = F_j^{-1} \left( \frac{p'_j - c_j - p'_i + \lambda'_j}{p'_j + \lambda'_j} \right) \).

**Proof.** The proof follows from the standard variational inequality theory (e.g., [1]).

4.2. Existence

Since the feasible set \( \Omega \) is not compact, we need to impose an additional condition to guarantee the existence of a solution.

Let \( \Omega_b \equiv \{(q_1, q_2, p, \rho) | 0 \leq (q_1, q_2) \leq b, (p, \rho) \in \Gamma \} \), where \( b = (b_1, b_2) \) and \( q_1 \leq b_1 \) and \( q_2 \leq b_2 \). \( \Omega_b \) is a bounded closed convex subset of \( \mathbb{R}^{I \times \Omega} \).

**Theorem 3.** (Existence). Suppose that there exist positive constants \( R \) and \( S \) such that

\[
\frac{\partial c_{ij}}{\partial q_{ij}} (q'_i) \geq R, \forall q \text{ with } q_i \geq S, \forall i, j.
\]

(12)

\[
\frac{\partial c_{ij}}{\partial q_{ij}} (q'_i) \geq R, \forall q \text{ with } q_i \geq S, \forall i, j.
\]

(13)

Then variational inequality (11) admits at least one solution.

**Proof.** The values of constants \( R \) and \( S \) are discussed in the existence proof in [50]. Following similar arguments on that proof, Assumptions (12) and (13) imply the existence of a constant \( b \) such that \((q_1, q_2) \leq b \) will guarantee the compactness of the set \( \Omega_b \) and therefore the existence of a solution of variational inequality (11). Assumptions (12) and (13) can be economically justified as follows. When the raw material shipment \( q_1 \) is large enough, one can expect the corresponding sum of the marginal costs associated with procurement and transaction to be large, which ensures (12). Similarly, when the product shipment \( q_2 \) is large, the corresponding sum of the marginal costs associated with production and transaction is expected to be large as well, which ensures (13).

4.3. Uniqueness

**Theorem 4.** (Uniqueness) Assume that cost functions, \( c_{ij} \) and \( c_{ij} \), are strictly convex and that the conditions in Theorem 1 are satisfied for each \( 1 \leq j \leq J \). Then variational inequality (11) admits a unique solution.

The proof is provided in Appendix B.

5. Examples of demand functions

[10] gives a detailed list of demand functions adopted in the literature and presents a survey of empirical evidence showing the application of these demand functions in real industry sectors (sugar, yogurt, peanut butter, fashion, retail). The following are examples of classical demand functions, outlined in [10] and included in our framework:

- **Additive Linear:**
  \( D_j(x) = x + a_j - b_j x + \sum_{k} c_k x_j, a_j > 0, c_k \geq 0, b_j > \sum_{k} c_k \) and \( A_j = a_j - b_j, c_k \geq 0, A_j > b_j \)

- **Multiplicative Isotonic (Power):**
  \( D_j(x) = a_j^{-b_j} \left( \prod_{k} x^k \right), a_j > 0, b_j > 1, c_k \geq 0 \)

- **Logit:**
  \( D_j(x) = a_j e^{-b_j x + \sum_{k} c_k x_j}, a_j > 0, c_k \geq 0, b_j > \sum_{k} c_k \)

- **Exponential:**
  \( D_j(x) = e^{a_j - b_j x + \sum_{k} c_k x_j}, a_j > 0, c_k \geq 0, b_j > \sum_{k} c_k \)

- **Logarithmic:**
  \( D_j(x) = \ln \left( x + a_j - b_j x + \sum_{k} c_k x_j \right), a_j > 0, c_k \geq 0, b_j > \sum_{k} c_k, A_j + a_j - b_j > \sum_{k} c_k x_j \)
• Logarithmic II:

\[
D_j(p, x) = \ln \left( \left( a_j - b_j p_j + \sum_{k \neq j} c_{jk} p_k \right)^x \right), \quad a_j > 0, \quad c_{jk} \geq 0, \quad b_j > \sum_{k \neq j} c_{jk} a_j - b_j p_j + \sum_{k \neq j} c_{jk} p_k > 1.
\]

• Mixed Additive-Multiplicative:

\[
D_j(p, x) = \mu_j(p) + \sigma_j(p) x, \quad \mu_j(p) + \sigma_j(p) + \lambda_j > 0.
\]

| \frac{\partial \mu_j(p)}{\partial p} | \leq 0, \quad \frac{\partial \sigma_j(p)}{\partial p} | \leq 0, \quad \forall k \neq j, \quad \frac{\partial \mu_j(p)}{\partial x} + \sum_{k \neq j} \frac{\partial \sigma_j(p)}{\partial x} \leq 0.
\]

| \frac{\partial \mu_j(p)}{\partial p} | \leq 0, \quad \frac{\partial \sigma_j(p)}{\partial p} | \leq 0, \quad \forall k \neq j, \quad \frac{\partial \mu_j(p)}{\partial p} + \sum_{k \neq j} \frac{\partial \sigma_j(p)}{\partial x} \leq 0.
\]

\[
\lambda_j(p, x) = \int_a^b \left( D_j(p, x) - D_j(p, x) \right) f_j(x) dx
\]

are the expected values of the leftover and shortage of retailer \( j \), respectively, and where \( D_j(p, x) = \sum_{i=1}^n q_i \) and \( x_j = \int_a^b \left( \frac{\lambda_j(x, x_j)}{\theta(x, x_j)} \right) f_j(x) dx \)

The following propositions illustrate the analytical effects of different types of model functions, model parameters (handling cost \( c_j \), shortage cost \( \lambda_j^s \) and salvage value \( \lambda_j^s \)), demand level, and demand variability on the optimal solutions and the expected profits. All proofs are given in Appendix C.

Proposition 1 Impact of the handling cost \( c_j \):

- The optimal quantities \( q_{ja}^* \) and \( q_{jb}^* \) decrease in \( c_j \).
- The optimal prices \( p_j^* \) increase in \( c_j \).
- The safety values \( z_j^* \) decrease in \( c_j \).
- The raw material suppliers’ profits \( \Pi_{ja}^b \), the manufacturers’ profits \( \Pi_{ja}^m \), the retailers’ profits \( \Pi_j \), and the total profit \( \Pi \) decrease in \( c_j \).

Proposition 2 Impact of the unit shortage cost \( \lambda_j^s \):

- If \( \frac{\partial \lambda_j(p, x)}{\partial p_j} \) is increasing in \( x \), the optimal quantities \( q_{ja}^* \) and \( q_{jb}^* \) increase in \( \lambda_j^s \).
- If \( \frac{\partial \lambda_j(p, x)}{\partial x} \) is decreasing in \( x \), the prices \( p_j^* \) increase in \( \lambda_j^s \).
- The safety values \( z_j^* \) increase in \( \lambda_j^s \).
- The total profit \( \Pi \) decrease in \( \lambda_j^s \).

Proposition 3 Impact of the unit salvage value \( \lambda_j^s \):

- If \( \frac{\partial \lambda_j(p, x)}{\partial p_j} \) is decreasing in \( x \), the optimal quantities \( q_{ja}^* \) and \( q_{jb}^* \) increase in \( \lambda_j^s \).
- If \( \frac{\partial \lambda_j(p, x)}{\partial x} \) is increasing in \( x \), the prices \( p_j^* \) increase in \( \lambda_j^s \).
- The safety values \( z_j^* \) increase in \( \lambda_j^s \).
- The total profit \( \Pi \) increase in \( \lambda_j^s \).

Proposition 4 Impact of demand level. Assume that for each retailer \( j \), the demand level is controlled by a parameter \( a_j \):

- The optimal quantities \( q_{ja}^* \) and \( q_{jb}^* \) decrease in \( a_j \).
- The prices \( p_j^* \) increase in \( a_j \).
Proposition 5 Impact of demand variability. Assume that for each retailer \( j \), the demand variability is controlled by a parameter \( m_j \).

- If the safety factors \( z^*_j \) are positive, the optimal quantities \( \tilde{q}^{\ast}_{in} \) and \( \tilde{q}^{\ast}_{io} \) increase in \( m_j \).
- If the safety factors \( z^*_j \) are negative, the optimal prices \( p^*_j \) decrease in \( m_j \).
- The total profit \( \Pi \) decreases in \( m_j \).

7. Numerical examples

A numerical study is carried out to show the effect of different model parameters on the equilibrium solution. As in [43], the extragradient algorithm of [55] is used to compute the solution of variational inequality (B.1). The algorithm is implemented in Matlab and has been successfully tested in previous studies [15,43].

After solving variational inequality (B.1), we can recover the equilibrium prices \( \bar{p}^{\ast}_{in} \) using the optimality conditions of variational inequality problem (2). If there is a positive shipment quantity \( \tilde{q}^{\ast}_{in} > 0 \), then \( \tilde{p}^{\ast}_{in} = \frac{2c_0}{m_0} \).

In our basic example, we consider a supply chain network with two raw material suppliers, two manufacturers and two retailers. The unit penalties of having excess supply/demand of retailers are set to \( \lambda_j = 2 \). \( \phi_j = 2 \) \( \forall j \). Also, the per-unit handling cost is set to \( c_0 = 40 \), \( \forall j \). The procurement and transaction cost functions faced by the suppliers and the production and transaction cost functions incurred by the manufacturers are given by:

\[
\begin{align*}
  c_{ni}(\tilde{q}_{in}) & = 1.5(\tilde{q}_{in})^2 + 10(\tilde{q}_{in}) + 2, \quad n = 1, i = 1, 2. \\
  c_{no}(\tilde{q}_{in}) & = 1.5(\tilde{q}_{in})^2 + 9(\tilde{q}_{in}) + 4, \quad n = 2, i = 1, 2. \\
  c_{ni}(\tilde{q}_{io}) & = 1.5(\tilde{q}_{io})^2 + 9(\tilde{q}_{io}) + 2, \quad i = 1, j = 1, 2. \\
  c_{no}(\tilde{q}_{io}) & = 1.5(\tilde{q}_{io})^2 + 11(\tilde{q}_{io}) + 2, \quad i = 2, j = 1, 2.
\end{align*}
\]

Four models will be considered for the demand functions at retailer outlets:

- Additive Linear (Model 1):
  \[
  D_j(p, e) = a_j - b_j p_j + \sum_{k \neq j} c_{jk} p_k + m_{jk},
  \]
  with \( (a_1, a_2) = (290, 300) \) and for all \( 1 \leq j \neq k \leq 2, b_j = 2 \) and \( c_{jk} = 1 \).

- Multiplicative Exponential (Model 2):
  \[
  D_j(p, e) = (m_j + e)^{a_j - b_j p_j + \sum_{k \neq j} c_{jk} p_k + m_{jk}},
  \]
  where \( (m_1, m_2) = (290, 300) \) and for all \( 1 \leq j \neq k \leq 2, a_j = 1, b_j = 0.02 \) and \( c_{jk} = 0.01 \).

- Mixed Linear-Exponential (Model 3):
  \[
  D_j(p, e) = a_j - b_j p_j + \sum_{k \neq j} c_{jk} p_k + e^{a_j - b_j p_j + \sum_{k \neq j} c_{jk} p_k + m_{jk}},
  \]
  where \( (a_1, a_2) = (290, 300) \) and for all \( 1 \leq j \neq k \leq 2, b_j = 2, c_{jk} = 1, a_j = 5, b_j = 0.02 \) and \( m_{jk} = 0.01 \).

- Logit(Model 4):
  \[
  D_j(p, e) = m_j \left( \frac{e^{a_j - b_j p_j + \sum_{k \neq j} c_{jk} p_k + m_{jk}} - 1}{e^{a_j - b_j p_j + \sum_{k \neq j} c_{jk} p_k + m_{jk}} + 1} \right).
  \]

where \( (m_1, m_2) = (100, 110) \) and for all \( 1 \leq j \neq k \leq 2, a_j = 5, b_j = 0.02 \) and \( c_{jk} = 0.01 \).

For the above four models, \( c_j \) for \( j = 1, 2 \), is chosen to have gamma distribution with shape parameter 2 and scale parameter 5. The distribution of \( c_j \) is centered and reduced to have a mean of 0 and variance 1. This is carried out to avoid over-parametrization, since each of the above models contains parameters that can be used to control demand average and variability. Table 1 displays the optimal equilibrium solutions, the expected profits of all raw material suppliers, manufacturers and retailers, and the total supply chain profit. Comparing the quantities \( \tilde{q}^{\ast}_{in} \) and \( \tilde{q}^{\ast}_{io} \), we observe lower values in the mixed linear-exponential model and higher values in the multiplicative exponential model. This is mainly due to the impact of the rate of decrease of the expected demand with respect to retailer prices. The faster the demand decreases with respect to the retailer price \( p_j \), the lower the quantities \( \tilde{q}^{\ast}_{io} \) retailers will order from manufacturers. A decrease in \( \tilde{q}^{\ast}_{io} \) will result in a decrease in the quantities \( \tilde{q}^{\ast}_{in} \), the expected profits \( \Pi^{\ast}_{in}, \Pi^{\ast}_{io} \) and \( \Pi^{\ast} \) and the total profit \( \Pi^{\ast} \). Note that based on the current shortage and salvage parameters, we obtain negative values of \( z^*_j \) in all four demand models, resulting in a situation that favors shortages at each retail outlet.

The impact of demand distributions on the equilibrium solutions was also tested using the mixed additive-multiplicative demand model (Model 3) with the parameters specified above. Table 2 illustrates the results for the uniform distribution \( U(-\sqrt{3}, \sqrt{3}) \), the reduced and truncated to \([-3, 3]\) normal distribution \( \mathcal{N}(0, 1) \), and the centered and reduced gamma distribution with parameters (2,5) as above. Comparing

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Equilibrium solutions for different demand models</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model</td>
<td>Model 1</td>
</tr>
<tr>
<td>( q_{i1} )</td>
<td>25.75</td>
</tr>
<tr>
<td>( q_{i2} )</td>
<td>26.47</td>
</tr>
<tr>
<td>( q_{1} )</td>
<td>25.42</td>
</tr>
<tr>
<td>( q_{2} )</td>
<td>26.14</td>
</tr>
<tr>
<td>( \bar{p}_{i1} )</td>
<td>25.95</td>
</tr>
<tr>
<td>( \bar{p}_{i2} )</td>
<td>25.61</td>
</tr>
<tr>
<td>( \bar{p}_1 )</td>
<td>26.28</td>
</tr>
<tr>
<td>( \bar{p}_2 )</td>
<td>25.95</td>
</tr>
<tr>
<td>( \bar{r}_i )</td>
<td>239.67</td>
</tr>
<tr>
<td>( \bar{r}_j )</td>
<td>242.53</td>
</tr>
<tr>
<td>( \bar{s}_i )</td>
<td>174.10</td>
</tr>
<tr>
<td>( \bar{s}_j )</td>
<td>176.25</td>
</tr>
<tr>
<td>( \bar{s}_1 )</td>
<td>87.84</td>
</tr>
<tr>
<td>( \bar{s}_2 )</td>
<td>86.84</td>
</tr>
<tr>
<td>( \bar{s}_3 )</td>
<td>87.84</td>
</tr>
<tr>
<td>( \bar{s}_4 )</td>
<td>86.84</td>
</tr>
<tr>
<td>( \bar{s}_5 )</td>
<td>1.00</td>
</tr>
<tr>
<td>( \bar{s}_6 )</td>
<td>1.00</td>
</tr>
<tr>
<td>( \Pi^{\ast}_{in} )</td>
<td>1296.34</td>
</tr>
<tr>
<td>( \Pi^{\ast}_{io} )</td>
<td>1370.20</td>
</tr>
<tr>
<td>( \Pi^{\ast} )</td>
<td>2039.95</td>
</tr>
<tr>
<td>( \Pi^{\ast} )</td>
<td>1988.06</td>
</tr>
<tr>
<td>( \Pi^{\ast} )</td>
<td>2037.65</td>
</tr>
<tr>
<td>( \Pi^{\ast} )</td>
<td>10721.96</td>
</tr>
</tbody>
</table>
The impact of demand level and demand variability are discussed in sections 6.2 and 6.3.

7.1. Impact of model parameters

Here, we illustrate numerically the impact of model parameters (handling cost $c_j$, shortage cost $\lambda_j^-$, and salvage value $\lambda_j^+$) on the optimal solutions and the expected profits.

7.1.1. Impact of the handling cost $c_j$

We first investigate the effect of changing the handling cost $c_j$ on the optimal quantities $\tilde{q}_{j\alpha}$ and $q_{j\beta}$, optimal prices $p_{j\alpha}$ and expected profits. The logit model (Model 4) is used in the illustration and similar results are expected for other demand models. As proven in Proposition 1, an increase in $c_j$ induces a decrease of the optimal quantities $\tilde{q}_{j\alpha}$ and $q_{j\beta}$, and safety values $z_j^-$ and an increase in the optimal prices $p_{j\beta}$ (see Table 3 for details). For the raw material suppliers, manufacturers, and retailers’ profits, as shown in Proposition 1, their expected profits decrease with $c_j$ which implies that the total profit decreases with $c_j$ as displayed in Table 3.

### Table 3
Impact of $c_j$ on quantity shipments, prices, safety values and expected profits (Model 4)

<table>
<thead>
<tr>
<th>$c_j$</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>80</th>
<th>90</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_{j1}$</td>
<td>28.17</td>
<td>27.39</td>
<td>26.59</td>
<td>25.79</td>
<td>24.98</td>
<td>24.15</td>
<td>23.32</td>
</tr>
<tr>
<td>$q_{j2}$</td>
<td>29.83</td>
<td>28.98</td>
<td>28.12</td>
<td>27.25</td>
<td>26.37</td>
<td>25.48</td>
<td>24.59</td>
</tr>
<tr>
<td>$q_{j3}$</td>
<td>27.84</td>
<td>27.06</td>
<td>26.26</td>
<td>25.46</td>
<td>24.64</td>
<td>23.82</td>
<td>22.99</td>
</tr>
<tr>
<td>$q_{j4}$</td>
<td>29.50</td>
<td>28.65</td>
<td>27.78</td>
<td>26.91</td>
<td>26.03</td>
<td>25.15</td>
<td>24.25</td>
</tr>
<tr>
<td>$q_{j5}$</td>
<td>28.84</td>
<td>28.02</td>
<td>27.19</td>
<td>26.35</td>
<td>25.51</td>
<td>24.65</td>
<td>23.79</td>
</tr>
<tr>
<td>$q_{j6}$</td>
<td>25.80</td>
<td>25.68</td>
<td>25.73</td>
<td>25.80</td>
<td>25.73</td>
<td>25.68</td>
<td>25.61</td>
</tr>
<tr>
<td>$q_{j7}$</td>
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<td>25.01</td>
<td>24.97</td>
<td>25.01</td>
<td>24.97</td>
<td>24.93</td>
<td>24.89</td>
</tr>
<tr>
<td>$q_{j8}$</td>
<td>25.84</td>
<td>25.02</td>
<td>24.97</td>
<td>25.01</td>
<td>24.97</td>
<td>24.93</td>
<td>24.89</td>
</tr>
<tr>
<td>$q_{j9}$</td>
<td>30.82</td>
<td>30.51</td>
<td>30.21</td>
<td>30.02</td>
<td>29.83</td>
<td>29.64</td>
<td>29.45</td>
</tr>
<tr>
<td>$q_{j10}$</td>
<td>299.32</td>
<td>301.59</td>
<td>303.87</td>
<td>306.17</td>
<td>308.47</td>
<td>310.79</td>
<td>313.11</td>
</tr>
<tr>
<td>$q_{j11}$</td>
<td>190.03</td>
<td>185.22</td>
<td>180.35</td>
<td>175.42</td>
<td>170.44</td>
<td>165.40</td>
<td>160.31</td>
</tr>
<tr>
<td>$q_{j12}$</td>
<td>96.51</td>
<td>94.05</td>
<td>91.57</td>
<td>89.05</td>
<td>86.52</td>
<td>83.95</td>
<td>81.36</td>
</tr>
<tr>
<td>$q_{j13}$</td>
<td>95.51</td>
<td>93.05</td>
<td>90.57</td>
<td>88.05</td>
<td>85.52</td>
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<td>80.36</td>
</tr>
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<td>91.57</td>
<td>89.05</td>
<td>86.52</td>
<td>83.95</td>
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</tr>
<tr>
<td>$q_{j15}$</td>
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<td>93.05</td>
<td>90.57</td>
<td>88.05</td>
<td>85.52</td>
<td>82.95</td>
<td>80.36</td>
</tr>
<tr>
<td>$z_1$</td>
<td>-0.76</td>
<td>-0.78</td>
<td>-0.81</td>
<td>-0.83</td>
<td>-0.85</td>
<td>-0.87</td>
<td>-0.89</td>
</tr>
<tr>
<td>$z_2$</td>
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7.1.2. Impact of the shortage cost \( \lambda_j \)

As proven in Proposition 2, the effect of the shortage cost \( \lambda_j \) on the optimal quantities \( q^*_j \) and \( q^*_ij \) and optimal prices \( p^*_j \) depends on the behavior of \( \frac{\partial (D(x))}{\partial \lambda_j} \). Two demand models (Model 3 and Model 4) are used to illustrate the different patterns. In fact, it can be seen that when \( \frac{\partial (D(x))}{\partial \lambda_j} \) is increasing in \( x \) (Model 4), the optimal quantities \( q^*_j \) and \( q^*_ij \) increase with \( \lambda_j \) as illustrated in Table 4. The sign of \( \frac{\partial (D(x))}{\partial \lambda_j} \) depends on the model parameters. In our illustration, the sign of \( \partial q^*_j/\partial \lambda_j \) depends on the value of \( \lambda_j \) as shown in Figure 2a (\( p^*_j \) increases for small \( \lambda_j \) and decreases for large \( \lambda_j \)). When \( \frac{\partial (D(x))}{\partial \lambda_j} \) decreases in \( x \) (Model 3), we find that \( \frac{\partial p^*_j}{\partial \lambda_j} \geq 0 \) and that the signs of \( \frac{\partial q^*_j}{\partial \lambda_j} \) and \( \frac{\partial q^*_ij}{\partial \lambda_j} \) depend on the model parameters. This behavior is illustrated in Table 5 and Figure 2b. Moreover, the safety value \( z^*_j \) increases with \( \lambda_j \) regardless of the demand function (Tables 4 and 5). Note that with large values of \( \lambda_j \), positive values of \( z^*_j \) are obtained implying the likelihood of oversupply at each retail outlet to cope with high shortage costs. For the expected profits of raw material suppliers and manufacturers, as discussed in C.2, \( \partial \Pi_m^s/\partial \lambda_j \) and \( \partial \Pi_m^m/\partial \lambda_j \) have the same sign as \( \partial q^*_j/\partial \lambda_j \) (Tables 4, and 5). On the other hand, the sign of \( \partial \Pi_m^r/\partial \lambda_j \) depends on the model parameters. In our illustration, for both Model 3 and 4 the expected profits of the retailers decrease with \( \lambda_j \) (Tables 4, and 5). For the total expected profit \( \Pi \), Proposition 2 shows that the total profit \( \Pi \) decreases with \( \lambda_j \) as displayed in Tables 4, and 5.

![Image](a) Selling prices \( p^*_j \) (Model 4)

![Image](b) Selling prices \( p^*_j \) (Model 3)
### Table 5

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Impact of \( \lambda_j^+ \) on the quantity shipments, prices, safety values and expected profits (Model 3)

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7.1.3. Impact of the salvage value \( \lambda_j^+ \)

The study of the effect of changing the salvage value \( \lambda_j^+ \) on the optimal quantities \( q_j^+ \) and \( q_j^- \) and optimal prices \( p_j \) is quite similar to that of the shortage value \( \lambda_j^- \). The same models (Models 3 and 4) are used for illustration purposes. As shown in Proposition 3, when \( \frac{\partial p_j(x, y)}{\partial y} \) increases in \( x \), the optimal prices \( p_j \) increase with \( \lambda_j^+ \) and the signs of \( \frac{\partial p_j(x, y)}{\partial y} \) and \( \frac{\partial q_j(x, y)}{\partial y} \) depend on the model parameters as illustrated in Table 6. When \( \frac{\partial p_j(x, y)}{\partial y} \) is decreasing in \( x \), we obtain \( \frac{\partial q_j(x, y)}{\partial y} \geq 0 \), \( \frac{\partial q_j(x, y)}{\partial y} > 0 \) and the sign of \( \frac{\partial q_j(x, y)}{\partial y} \) depends on the model parameters as displayed in Table 7. Moreover, the safety values \( s_j \) increase with \( \lambda_j^+ \) regardless of the demand function (Tables 6 and 7). For the expected profits, Proposition 3 has proved that the total profit increases with \( \lambda_j^+ \) as displayed in Tables 6 and 7. As discussed in C.3, \( \frac{\partial \Pi_j}{\partial \lambda_j^+} \) and \( \frac{d\Pi_j}{\partial \lambda_j^+} \) have the same sign as \( \frac{\partial q_j(x, y)}{\partial y} \) as illustrated in Tables 6 and 7.

For the expected profits of the retailers, the sign of \( \frac{\partial \Pi_j}{\partial \lambda_j^+} \) can be positive or negative depending on the model parameters. Figure 3a shows the case when the retailers’ expected profits increase with \( \lambda_j^+ \) (Model 4) while Figure 3b shows the case when the retailers’ expected profits decrease with \( \lambda_j^+ \) (Model 3).

7.2. Impact of demand level

The effect of demand level on the optimal quantity shipments, prices, and profits is explored. As mentioned in C.4, the mixed additive-
Table 8

Impact of $q_j$ on the quantity shipments, prices, safety values and expected profits

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<td>27.49</td>
<td>28.73</td>
<td>29.95</td>
<td>31.16</td>
</tr>
<tr>
<td>$q_{12}$</td>
<td>24.40</td>
<td>25.67</td>
<td>26.92</td>
<td>28.16</td>
<td>29.39</td>
<td>30.61</td>
<td>31.83</td>
</tr>
<tr>
<td>$q_{21}$</td>
<td>24.07</td>
<td>25.33</td>
<td>26.58</td>
<td>27.83</td>
<td>29.06</td>
<td>30.28</td>
<td>31.50</td>
</tr>
</tbody>
</table>

Table 9

Impact of $m_j$ on the quantity shipments, prices, safety values and expected profits ($j^* = 2$)

<table>
<thead>
<tr>
<th>$m_j$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_{11}$</td>
<td>25.75</td>
<td>25.64</td>
<td>25.53</td>
<td>25.42</td>
<td>25.31</td>
<td>25.20</td>
<td>25.09</td>
</tr>
<tr>
<td>$q_{12}$</td>
<td>26.47</td>
<td>26.36</td>
<td>26.25</td>
<td>26.14</td>
<td>26.03</td>
<td>25.92</td>
<td>25.81</td>
</tr>
<tr>
<td>$q_{21}$</td>
<td>25.42</td>
<td>25.31</td>
<td>25.20</td>
<td>25.09</td>
<td>24.98</td>
<td>24.87</td>
<td>24.76</td>
</tr>
<tr>
<td>$q_{22}$</td>
<td>26.14</td>
<td>26.03</td>
<td>25.92</td>
<td>25.81</td>
<td>25.69</td>
<td>25.58</td>
<td>25.47</td>
</tr>
<tr>
<td>$q_1$</td>
<td>25.95</td>
<td>25.84</td>
<td>25.72</td>
<td>25.61</td>
<td>25.50</td>
<td>25.39</td>
<td>25.28</td>
</tr>
<tr>
<td>$q_2$</td>
<td>25.61</td>
<td>25.50</td>
<td>25.39</td>
<td>25.28</td>
<td>25.17</td>
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<td>26.06</td>
<td>25.95</td>
<td>25.84</td>
<td>25.73</td>
<td>25.61</td>
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<tr>
<td>$q_{21}$</td>
<td>25.95</td>
<td>25.84</td>
<td>25.72</td>
<td>25.61</td>
<td>25.50</td>
<td>25.39</td>
<td>25.28</td>
</tr>
</tbody>
</table>

Operations Research Perspectives 7 (2020) 100165
multiplicative model (Model 3) with $\mu(p) = a_0 - b_0 p + \sum_k c_k p_k$ is used for illustration. The parameter $a_0$ is used to control the demand level. From Proposition 4, the optimal quantities $q_{ij}^*$ and $q_{ij}'$ and the optimal price $p_j^*$ increase with $a_0$ as illustrated in Table 8. Note that the safety values $x_j^*$ decrease with $a_0$ (Table 8). Additionally, as proven in Proposition 4, the expected profits $\Pi_0, \Pi_0^r, \Pi_0^w,$ and $\Pi$ increase with $a_0$ as displayed in Table 8. Note that based on the model parameters, the expected profits of retailers also increase with $a_0$.

7.3. Impact of demand variability

The effect of demand variability on the optimal equilibrium is explored. The linear additive model (Model 1) with $D_j(p, x) = a_j - b_j p_j + \sum_{k,j'} c_{jk} p_k + m_j x$ is used for this analysis and $m_j$ is used to control demand variability. From Proposition 5, it can be seen that $\frac{\partial q_{ij}^*}{\partial a_0} \leq 0$ and $\frac{\partial q_{ij}'}{\partial a_0} \leq 0$ when $x^*$ is positive and their sign depend on the model parameters when $x_j$ is negative. For the optimal prices, $\frac{\partial q_{ij}^*}{\partial a_0} \leq 0$ when $x_j^*$ is negative and its sign depends on the model parameters when $x_j$ is positive. In our setting, the optimal quantities $q_{ij}^*$ and $q_{ij}'$ decrease with $m_j$ when $x_j^*$ is negative (Table 9). When $x_j$ is positive, the optimal quantities $q_{ij}^*$ and $q_{ij}'$ increase and the optimal prices $p_j^*$ decrease with $m_j$ (Table 10). For the expected profits, Proposition 5 shows that the total profit $\Pi$ decreases with $m_j$ as displayed in Tables 9 and 10. Since the sign of $\frac{\partial \Pi}{\partial m_j}$ and $\frac{\partial \Pi^r}{\partial m_j}$ have the same sign as $\frac{\partial q_j^*}{\partial m_j}$, the profits of the raw material suppliers and manufacturers decrease with $m_j$ when $x_j^*$ is negative (Table 9) and increase with $m_j$ when $x_j$ is positive (Table 10). Consequently, a positive value of $x_j$ will induce retailers to order more from manufacturers whos profit from demand uncertainty when $x_j > 0$. Note that for both cases (negative and positive values of $x_j$), the expected profits of the retailers decrease with $m_j$ (Tables 9 and 10).

8. Managerial insights

The summary of our key findings from the sensitivity analysis and numerical tests are as follows.

- The effect of model parameters on the equilibrium solutions depends on the type of demand model. For example, the effect of the shortage cost $\lambda_j$ depends on the behavior of $\frac{\partial (x_j)}{\partial x_j}$. For the logit model (Model 4), $\frac{\partial (\lambda)}{\partial x_j}$ is increasing in $x$ and the optimal quantities $q_{ij}^*$ and $q_{ij}'$ increase with $x_j$ as illustrated in Table 4. The increase of $q_{ij}^*$ and $q_{ij}'$ will induce an increase of the expected profits of raw material suppliers and manufacturers (Table 4). On the other hand, using the mixed linear-exponential model (Model 3), $\frac{\partial (\lambda)}{\partial x_j}$ is decreasing in $x$ and the optimal quantities $q_{ij}^*$ and $q_{ij}'$ decrease with $x_j$ resulting in a decrease of the expected profits of raw material suppliers and manufacturers (Table 5). Another example is the effect of the salvage value $\lambda_j^s$. With the logit model (Model 4), the retailers’ expected profits increase with $\lambda_j^s$ as shown in Table 6. However, using the mixed linear-exponential model (Model 3), the retailers’ expected profits decrease with $\lambda_j^s$ as illustrated in Table 7.
- For the same demand model, the effect of demand variability on the various supply chain members depends on the model parameters. Taking the linear additive model (Model 1) as an example, the total

| Table 10 Impact of $m_j$ on the quantity shipments, prices, safety values and expected profits ($x_j^* = 300$) |
|---|---|---|---|---|---|---|---|
| $m_j$ | 2   | 3   | 4   | 5   | 6   | 7   | 8   |
| $q_{11}$ | 26.03 | 26.05 | 26.07 | 26.10 | 26.12 | 26.15 | 26.17 |
| $q_{13}$ | 25.69 | 25.72 | 25.74 | 25.77 | 25.79 | 25.81 | 25.84 |
| $q_{16}$ | 25.88 | 25.91 | 25.93 | 25.95 | 25.98 | 25.99 | 26.02 |
| $q_{17}$ | 26.55 | 26.57 | 26.60 | 26.62 | 26.64 | 26.65 | 26.69 |
| $p_1$ | 241.19 | 241.16 | 241.13 | 241.10 | 241.07 | 241.04 | 241.01 |
| $p_2$ | 244.04 | 244.01 | 243.98 | 243.95 | 243.92 | 243.89 | 243.85 |
| $\pi_1$ | 175.73 | 175.87 | 176.02 | 176.16 | 176.29 | 176.43 | 176.58 |
| $\pi_2$ | 177.86 | 178.01 | 178.15 | 178.29 | 178.43 | 178.57 | 178.71 |
| $\Pi_0$ | 88.65 | 88.72 | 88.79 | 88.86 | 88.93 | 89.00 | 89.07 |
| $\Pi_0^r$ | 87.65 | 87.72 | 87.79 | 87.86 | 87.93 | 88.00 | 88.07 |
| $\Pi_0^w$ | 88.65 | 88.72 | 88.79 | 88.86 | 88.93 | 89.00 | 89.07 |
| $\Pi_0^s$ | 87.65 | 87.72 | 87.79 | 87.86 | 87.93 | 88.00 | 88.07 |
| $\Pi_1$ | 0.03 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 |
| $\Pi_2$ | 0.03 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 |
| $\Pi_0^{\prime}$ | 902.88 | 688.26 | 473.68 | 259.15 | 44.67 | -169.77 | -384.15 |
| $\Pi_1^{\prime}$ | 975.22 | 759.39 | 543.61 | 327.88 | 112.19 | -103.27 | -319.03 |
| $\Pi_0^{\prime r}$ | 2082.49 | 2086.32 | 2090.13 | 2093.91 | 2097.65 | 2101.36 | 2105.05 |
| $\Pi_1^{\prime r}$ | 2030.05 | 2033.84 | 2037.60 | 2041.33 | 2045.03 | 2048.69 | 2052.33 |
| $\Pi_0^{\prime w}$ | 2031.76 | 2035.55 | 2039.31 | 2043.03 | 2046.73 | 2050.40 | 2054.04 |
| $\Pi_1^{\prime w}$ | 2080.19 | 2084.03 | 2087.83 | 2091.61 | 2095.36 | 2099.07 | 2102.75 |
| $\Pi$ | 10102.59 | 9687.39 | 9272.16 | 8856.91 | 8441.62 | 8026.32 | 7610.99 |
expected profit is always decreasing with the demand variability. We expect the same behavior with the profits of retailers who are directly affected by an increase of demand variability (Table 9).

However, the behavior of the expected profits of raw material suppliers and manufacturers depends on the model parameters. As illustrated in Table 9, the profits of the raw material suppliers and manufacturers decrease with the demand variability when the optimal safety values \( z_j^* \) are negative. When \( z_j^* \) are positive, the profits of the raw material suppliers and manufacturers increase with the demand variability (Table 10). Consequently, a positive value of \( z_j^* \) will induce retailers to order more from manufacturers and manufacturers to order more from raw material suppliers implying that both raw material suppliers and manufacturers would profit from demand uncertainty when \( z_j^* > 0 \).

Clearly, the results and insights obtained in our paper illustrate the importance of identifying the type of demand model in practice and generate interesting practical implications for managers and decision makers. First, the effect of a change in model parameters, like shortage cost and salvage value, on supply chain members is not straightforward and might result in different behaviors depending on the type of demand model. Therefore, all supply chain members should seek knowledge of the type of consumer demand model in their setting. Second, the effect of demand variability is only obvious in the case of the retailers who loose from an increased demand variability but might be counter intuitive for other supply chain members. In particular, depending on whether retailers best choice involves overstocking or not, manufacturers and raw material suppliers can either profit or loose from an increased demand variability. Finally, our new model can assist supply chain operations managers to quantify the effects of different types of demand functions, model parameters, demand level, and demand variability on quantity shipments, prices, and expected profits.

9. Conclusion

The concept of supply chain equilibrium has received increased attention in the supply chain management literature. Our study contributes to research in supply chain equilibrium by providing insights on how the type of demand function and model parameters affect the decisions and performance of the supply chain. In this paper, we develop a new supply chain equilibrium model in a network consisting of multiple suppliers, manufacturers and retailers who sell the product directly in their own demand markets. Demand uncertainty is modeled using a general demand model including additive, multiplicative, power, and logit functions. Moreover, to account for competitiveness, the demand for the product at each retail outlet is price-sensitive and depends on all retail prices.

Using a variational inequality approach, we derive the equilibrium conditions of raw material suppliers, manufacturers, and retailers. Existence and uniqueness of the equilibrium quantities and prices are discussed and an extragradient-based algorithm is proposed to solve the model. Sensitivity analysis and numerical examples illustrate the flexibility of the model and show the impact of demand function, model parameters, demand level and demand variability on the equilibrium shipments, prices, and expected profits.

Our model establishes the foundation for supply chain equilibrium problems under general price-dependent demand. The model is limited to a one-period setting with only two stages in the supply chain network. Additionally, our model does not consider capacity constraints and correlation of the retailers’ uncertainties. Future research could extend the model to address these limitations. Other interesting avenues of future research include modeling the supply chain problem under general price-dependent demand within a Stackelberg equilibrium game framework and/or examining different type of contracts and incentives that could lead to supply chain coordination in networks with general demand functions.

Declaration of Competing Interest

The authors whose names are listed immediately below certify that they have NO affiliations with or involvement in any organization or entity with any financial interest (such as honoraria; educational grants; participation in speakers bureaus; membership, employment, consultancies, stock ownership, or other equity interest; and expert testimony or patent-licensing arrangements), or non-financial interest (such as personal or professional relationships, affiliations, knowledge or beliefs) in the subject matter or materials discussed in this manuscript.

Appendix A. Proof of Theorem 1

To prove Theorem 1, note that (8) yields

\[
\frac{\partial \Pi}{\partial p_j} = \left( p_j + \lambda_j - \lambda_j^* \right) \int_A^B \frac{\partial D_j(p, x)}{\partial p_j} f_j(x) \, dx - \int_A^B \frac{\partial p_j}{\partial p_j} f_j(x) \, dx + \int_A^B D_j(p, x) f_j(x) \, dx + D_j \left(p, \tau_j^* \right) \left(1 - F_j \left(\tau_j^* \right) \right),
\]

(A.1)

where \( \bar{F}_j = F_j \left( \frac{p - \lambda_j - \lambda_j^*}{\lambda_j} \right) \). Using integration by parts and the definition of \( E_j(p, x) \), \( \frac{\partial \Pi}{\partial p_j} \) can be rewritten as:

\[
\frac{\partial \Pi}{\partial p_j} = \int_A^B \left[ 1 - \frac{p_j - \lambda_j^*}{\lambda_j} \right] \frac{\partial D_j(p, x)}{\partial x} \left(1 - F_j(x) \right) \, dx + \int_A^B \frac{\lambda_j^*}{\lambda_j} \frac{\partial D_j(p, x)}{\partial x} \left(1 - F_j(x) \right) \, dx + D_j(p, A_j)
\]

(A.2)

\[
= \int_A^B \left[ 1 - \frac{p_j + \lambda_j - \lambda_j^*}{\lambda_j} \right] \frac{\partial D_j(p, x)}{\partial x} \left(1 - F_j(x) \right) \, dx + \int_A^B \frac{\lambda_j^*}{\lambda_j} \frac{\partial D_j(p, x)}{\partial x} \left(1 - F_j(x) \right) \, dx + D_j(p, A_j)
\]

(A.3)

Using (A.1), (A.2) and (A.3), it can be seen that

\[
\frac{\partial \Pi}{\partial p_j} = 0 \iff \left( p_j + \lambda_j - \lambda_j^* \right) \int_A^B \frac{\partial D_j(p, x)}{\partial p_j} f_j(x) \, dx - \lambda_j \int_A^B \frac{\partial p_j}{\partial p_j} f_j(x) \, dx = - \int_A^B D_j(p, x) f_j(x) \, dx + D_j \left(p, \tau_j^* \right) \left(1 - F_j \left(\tau_j^* \right) \right).
\]

(A.4)
\[
\frac{\partial \Pi}{\partial p_j} = 0 \iff \int_{\alpha_j}^{\gamma_j} \left[ 1 - \frac{p_j + \lambda_j^- - \lambda_j^+}{p_j} \right] \frac{\partial D_j(p, x)}{\partial x} (1 - F_j(x)) dx \leq 0.
\]
\[
\frac{\partial \Pi}{\partial p_j} = 0 \iff \int_{\alpha_j}^{\gamma_j} \left[ 1 - \frac{p_j + \lambda_j^- - \lambda_j^+}{p_j} \right] \frac{\partial D_j(p, x)}{\partial x} (1 - F_j(x)) dx = -\int_{\alpha_j}^{\gamma_j} \frac{\lambda_j^-}{p_j} \frac{\partial \mathcal{E}_j(p, x)}{\partial x} \frac{\partial D_j(p, x)}{\partial x} (1 - F_j(x)) dx - D_j(p, A_j).
\]

Since \( \mathcal{E}_j(p, x) \) is increasing in \( x \) (Assumption 2.iii) and by (A.6),
\[
\int_{\alpha_j}^{\gamma_j} \left[ 1 - \frac{p_j + \lambda_j^- - \lambda_j^+}{p_j} \right] \frac{\partial D_j(p, x)}{\partial x} (1 - F_j(x)) dx \leq 0.
\]

we get \( \frac{p_j + \lambda_j^- - \lambda_j^+}{p_j} \mathcal{E}_j(p, x) \geq 1 \), which is equivalent to
\[
\frac{\partial D_j(p, \tau_j)}{\partial p_j} \left( 1 - F_j(\tau_j) \right) + \frac{\partial D_j(p, \tau_j)}{\partial x} \left( 1 - F_j(\tau_j) \right)^2 \leq 0.
\]

From (A.1) and (A.2), the second derivative of \( \Pi \) with respect to \( p_j \) can be calculated in two ways:
\[
\frac{\partial^2 \Pi}{\partial p_j^2} = \int_{\alpha_j}^{\gamma_j} \left( p_j + \lambda_j^- - \lambda_j^+ \right) \frac{\partial^2 D_j(p, x)}{\partial x^2} f_j(x) dx - \lambda_j^- \int_{\alpha_j}^{\gamma_j} \frac{\partial^2 D_j(p, x)}{\partial p_j^2} f_j(x) dx
\]
\[
+ 2 \int_{\alpha_j}^{\gamma_j} \frac{\partial D_j(p, x)}{\partial p_j} \frac{\partial D_j(p, \tau_j)}{\partial p_j} \left( 1 - F_j(\tau_j) \right) + \frac{\partial D_j(p, \tau_j)}{\partial x} \left( 1 - F_j(\tau_j) \right)^2 \leq 0.
\]

or
\[
\frac{\partial^2 \Pi}{\partial p_j^2} = \int_{\alpha_j}^{\gamma_j} \left[ 1 - \frac{p_j + \lambda_j^- - \lambda_j^+}{p_j} \right] \frac{\partial \mathcal{E}_j(p, x)}{\partial x} \frac{\partial D_j(p, x)}{\partial x} (1 - F_j(x)) dx + \int_{\alpha_j}^{\gamma_j} \frac{\lambda_j^-}{p_j} \frac{\partial \mathcal{E}_j(p, x)}{\partial x} \frac{\partial D_j(p, x)}{\partial x} (1 - F_j(x)) dx
\]
\[
+ \frac{\partial^2 D_j(p, x)}{\partial p_j \partial x} \left[ 1 - \frac{p_j + \lambda_j^- - \lambda_j^+}{p_j} \right] \frac{\partial D_j(p, x)}{\partial x} (1 - F_j(x)) dx
\]
\[
+ \frac{\partial^2 D_j(p, x)}{\partial p_j^2} \left[ 1 - \frac{p_j + \lambda_j^- - \lambda_j^+}{p_j} \right] \frac{\partial D_j(p, x)}{\partial x} (1 - F_j(x)) dx + \frac{\partial D_j(p, A_j)}{\partial p_j} + \frac{\partial \Pi}{\partial p_j} \left( 1 - F_j(\tau_j) \right) + \frac{\partial D_j(p, \tau_j)}{\partial x} \left( 1 - F_j(\tau_j) \right)^2 \leq 0.
\]
A1. First case: $\frac{\partial D_i(p,x)}{\partial \lambda} \leq 0$

In this case, $\frac{\partial D_i(p,x)}{\partial \lambda} = A1 + A2 + A3$, where $A1$, $A2$, and $A3$ are given by the respective terms in (A.8), (A.9) and (A.10). Because $\frac{\partial D_i(p,x)}{\partial \lambda} \leq 0$ and $\frac{\partial D_i(p,x)}{\partial \lambda} \leq \frac{1}{p} \frac{\partial (\lambda j_j)}{\partial \lambda_j}$, terms $A2$ and $A3$ are nonpositive. Note that when the shortage cost $\lambda_j = 0$, it is easy to show that $A1 \leq 0$ since $\frac{\partial D_i(p,x)}{\partial \lambda} \leq 0$. For $\lambda_j > 0$, the argument used to show that $A1 \leq 0$ depends on which part of Assumption 3 is satisfied. The details are outlined next in three subsections.

A1.1. $D_j(p,x)$ satisfies Assumption 3.i

In this case, $\frac{\partial D_j(p,x)}{\partial \lambda}$ is increasing in $x$, then

$$A1 = \left( p_j - \lambda_j \right) \int_0^\lambda \frac{\partial D_j(p,x)}{\partial \lambda} f_j(x) dx - \lambda_j \int_\lambda^0 \frac{\partial D_j(p,x)}{\partial \lambda} f_j(x) dx$$

$$\leq \frac{\partial D_j(p,z_j)}{\partial \lambda} \left[ \left( p_j + \lambda_j - \lambda_j \right) F_j \left( \lambda_j \right) - \lambda_j \right] = \frac{\partial D_j(p,z_j)}{\partial \lambda} \left[ p_j - \lambda_j - \lambda_j \right] \leq 0.$$

The last equality holds because of equation (A.4).

A1.2. $D_j(p,x)$ satisfies Assumption 3.ii

In this case, $\frac{\partial D_j(p,x)}{\partial \lambda}$ is increasing in $x$, then

$$A1 \leq \left( p_j - \lambda_j \right) \int_0^\lambda \frac{\partial D_j(p,x)}{\partial \lambda} f_j(x) dx + \lambda_j \int_\lambda^0 \frac{\partial D_j(p,x)}{\partial \lambda} f_j(x) dx$$

$$\leq \frac{\partial D_j(p,z_j)}{\partial \lambda} \left[ \int_0^\lambda D_j(p,x)f_j(x) dx + D_j(p,z_j) \left( 1 - F_j \left( \lambda_j \right) \right) \right] \leq 0.$$

A1.3. $D_j(p,x)$ satisfies Assumption 3.iii

In this case, $\frac{\partial D_j(p,x)}{\partial \lambda}$ is increasing in $x$, then

$$A1 \leq \frac{\partial D_j(p,z_j)}{\partial \lambda} \left( p_j + \lambda_j - \lambda_j \right) D_j(p,z_j)f_j(z_j) \frac{\partial \xi_j}{\partial \lambda} - \int_\lambda^0 D_j(p,x)f_j(x) dx$$

$$= \left( p_j + \lambda_j - \lambda_j \right) D_j(p,z_j)f_j(z_j) \frac{1 - F_j \left( \lambda_j \right)}{\left( p_j + \lambda_j - \lambda_j \right) f_j \left( \lambda_j \right)} - \int_\lambda^0 D_j(p,x)f_j(x) dx$$

$$\leq D_j(p,z_j) \left( 1 - F_j \left( \lambda_j \right) \right) - D_j(p,z_j) \left( 1 - F_j \left( \lambda_j \right) \right) = 0.$$

Consequently, there exists $\lambda_j^0 > 0$ such that $H(\lambda_j^0) > 0$ for each $0 \leq \lambda_j \leq \lambda_j^0$, which implies that $A1 \leq 0$ for all these values of $\lambda_j$.

A2. Second case: $\frac{\partial D_i(p,x)}{\partial \lambda} \geq 0$

According to Assumption 3.iv, $\frac{\partial D_i(p,x)}{\partial \lambda} \leq 0$, $\frac{\partial D_i(p,x)}{\partial x}$ is decreasing in $x$ and $\frac{\partial D_i(p,x)}{\partial \lambda}$ is independent of $x$. In this case, we have $\frac{\partial D_i(p,x)}{\partial \lambda} = B1 + B2 + B3 + B4$ where $B1$, $B2$, $B3$, and $B4$ are given by the respective terms in (A.11),
(A.12), (A.13) and (A.14). Because \( \frac{d\phi(p,x)}{dx} \geq 0 \) and \( \frac{p}{p_j} - \frac{\lambda_j^+}{p_j} \geq 1 \), terms B2 and B4 are nonpositive. Since \( \frac{\phi(p,x)}{p_j} \) is decreasing in \( x \), we get:

\[
B1 = \int_{\lambda_j}^{x_j} - \frac{p_j - \lambda_j^+}{p_j} \frac{d\phi(p,x)}{dx} \left( 1 - F_j(x) \right) dx + \int_{\lambda_j}^{x_j} \frac{\lambda_j^+}{p_j} \frac{d\phi(p,x)}{dx} \left( 1 - F_j(x) \right) dx \]

Consequently, \( B1 \leq \frac{\phi(p,x)}{p_j} = \frac{\phi(p,x)}{p_j} \) because of equation (A.5).

The last equality holds because of equation (A.5).

Now, because \( \frac{\phi(p,x)}{p_j} \) is independent of \( x \), we obtain:

\[
B3 = \int_{\lambda_j}^{x_j} \frac{\phi(p,x)}{p_j} \left( 1 - \frac{p_j - \lambda_j^+}{p_j} \frac{\phi(p,x)}{p_j} \right) \left( 1 - F_j(x) \right) dx + \int_{\lambda_j}^{x_j} \frac{\phi(p,x)}{p_j} \frac{\lambda_j^+}{p_j} \left( 1 - F_j(x) \right) dx \]

The last equality holds because of equation (A.5).

Since \( \eta_j(p,x) \) is decreasing in \( x \) (Assumption 2.ii), we have \( \frac{\phi(p,x)}{p_j} \leq \frac{\phi(p,x)}{p_j} \) which implies that

\[
B3 \leq - \frac{\phi(p,x)}{p_j} D_j(p,A_j) + \frac{\phi(p,x)}{p_j} \frac{\phi(p,x)}{p_j} = 0.
\]

Consequently, \( \frac{\phi(p,x)}{p_j} \leq 0 \) in assumptions 3.i), 3.ii), 3.iii) and 3.iv) and therefore \( \Pi_j \) is pseudo-concave in \( p_j \).

Appendix B. Proof of Theorem 4

Variational inequality (11) can be rewritten in standard form as follows: determine \( X' \in \Omega \) such that

\[
\langle \mathcal{F}(X'), X - X' \rangle \geq 0, \quad \forall X \in \Omega,
\]

where \( X \equiv (q_1, q_2, p, \rho) \) and \( \mathcal{F}(X) \equiv (\mathcal{F}_w, \mathcal{F}_y, \mathcal{F}_j, \mathcal{F}_j) \), with the specific components of \( \mathcal{F}(X) \) being given by the respective functional terms preceding the multiplication signs in (11):
\[
\mathcal{F}_u(q_1, q_2, p, \rho) = \frac{\partial \mathcal{u}}{\partial q_u}, \\
\mathcal{F}_\rho(q_1, q_2, p, \rho) = \frac{\partial \mathcal{u}}{\partial \rho} - \rho; \\
\mathcal{F}_\rho^{(1)}(q_1, q_2, p, \rho) = \frac{\partial \Pi}{\partial \rho} = -\int_0^1 D_1 \left( p(x) f_1(x) dx - D_1 \left( p(z_i) \right) \right), \\
\mathcal{F}_\rho^{(2)}(q_1, q_2, p, \rho) = \frac{\partial \Pi^{(1)}}{\partial \rho} = -\left( \rho_i + \lambda_i^2 - \lambda_j^2 \right) \int_0^1 \frac{\partial D_1(p(x))}{\partial \rho} f_1(x) dx + \lambda_j \int_0^1 \frac{\partial D_1(p(x))}{\partial \rho} f_1(x) dx \\
\mathcal{F}_\rho^{(3)}(q_1, q_2, p, \rho) = \frac{\partial \Pi^{(2)}}{\partial \rho} = \sum_{i=1}^n q_i - D_1(p(z_i)) \text{ where } z_i = F^{-1}_1 \left( \frac{p_i - c_i - \rho_i + \lambda_i^2}{\rho_i + \lambda_i^2 - \lambda_j^2} \right) \\
\Phi \leq k_1 \left( D_1, D_2, D_3, D_4 \right), \quad \text{where } A_1^3 \text{ is a } J \times J \text{ diagonal matrix with } \left( A_1^3 \right)_{ij} = \frac{\partial^3 \mathcal{u}}{\partial q_i \partial q_j \partial q_k} \left( 1 \leq i \leq J \right), I_{J \times J} \text{ is the identity matrix with rank } J, O_{J \times J} \text{ and } O_{J} \text{ are } I \times I \text{ and } J \times J \text{ matrices of zeros, and matrices } D_1, D_2, D_3 \text{ and } D_4 \text{ are calculated as:} \\
D_1 = \begin{pmatrix} \frac{\partial^3 \Pi_{11}}{\partial q_1 \partial q_2 \partial q_1} & \frac{\partial^3 \Pi_{12}}{\partial q_1 \partial q_2 \partial q_2} & \cdots & \frac{\partial^3 \Pi_{1J}}{\partial q_1 \partial q_2 \partial q_J} \\
\frac{\partial^3 \Pi_{21}}{\partial q_1 \partial q_2 \partial q_2} & \frac{\partial^3 \Pi_{22}}{\partial q_1 \partial q_2 \partial q_2} & \cdots & \frac{\partial^3 \Pi_{2J}}{\partial q_1 \partial q_2 \partial q_J} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^3 \Pi_{J1}}{\partial q_1 \partial q_2 \partial q_2} & \frac{\partial^3 \Pi_{J2}}{\partial q_1 \partial q_2 \partial q_2} & \cdots & \frac{\partial^3 \Pi_{JJ}}{\partial q_1 \partial q_2 \partial q_J} \\
\end{pmatrix}, \quad D_2 = \begin{pmatrix} \gamma_1 & 0 & \cdots & 0 \\
0 & \gamma_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \gamma_J \\
\end{pmatrix}, \\
D_3 = \begin{pmatrix} -\gamma_1 & -\gamma_2 & \cdots & -\gamma_J \\
-\gamma_1 & -\gamma_2 & \cdots & -\gamma_J \\
\vdots & \vdots & \ddots & \vdots \\
-\gamma_1 & -\gamma_2 & \cdots & -\gamma_J \\
\end{pmatrix}, \quad \text{and } D_4 = \begin{pmatrix} \beta_1 & 0 & \cdots & 0 \\
0 & \beta_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \beta_J \\
\end{pmatrix} \text{ with } \beta_i = \frac{\partial^3 \mathcal{u}}{\partial q_i \partial q_j \partial q_k} \text{ and } \gamma_j = \frac{\partial \Pi}{\partial q_j} + \frac{\partial \Pi}{\partial q_k} \left( 1 - F^{-1}(p) \right) \frac{\partial F^{-1}(p)}{\partial q_j}. \\
\end{align*}

In order for the equilibrium vector to be unique, Theorem 1 of [49] requires \( \det(M)_{|\mathcal{X} = \mathcal{X}'} > 0 \) for all equilibria \( \mathcal{X}' \), along with two additional minor conditions: differentiability and boundary requirements. The former condition is met as assumed, and the boundary condition was needed to prove the existence of equilibrium. To prove the uniqueness of the equilibrium solution, it remains to be shown that \( \det(M)_{|\mathcal{X} = \mathcal{X}'} > 0 \).

Since functions \( \mathcal{u}_1 \) and \( \mathcal{u}_2 \) are assumed to be strictly convex, all matrices \( A_1^3 \) and \( A_2^3 \) are invertible with positive determinants and therefore matrix \( A \) is also invertible with \( \det(A) > 0 \). Consequently, \( \det(M) = \det(D - CA^{-1}B) \det(A) \). Using the definitions of matrices \( B \) and \( C \), it can be seen that \( -CA^{-1}B = \begin{pmatrix} O_{J \times J} & O_{J} \end{pmatrix} \), where \( U \) is a \( J \times J \) diagonal matrix with \( U_{ij} = \sum_{l=1}^{J} \frac{1}{\mathcal{u}_l}. \) Therefore, \( D - CA^{-1}B = \begin{pmatrix} D_1 & D_2 \\
D_3 & D_4 + U \end{pmatrix} \). Since matrix \( D_4 + U \) is a diagonal matrix with positive elements equal to \( \delta_j = \beta_j + U_{jj}, \) then \( \det(D - CA^{-1}B) = \det(D_1 - D_2(D_4 + U)^{-1}D_3) \det(D_4 + U) \). Using the definitions of matrices \( D_1, D_2, D_3, \) and \( D_4 + U \), it can be seen that the elements of the matrix \( N = D_1 - D_2(D_4 + U)^{-1}D_3 \) are given by \( N_{ij} = \frac{\partial \Pi}{\partial q_i} + \frac{\partial \Pi}{\partial q_j} \) \( \forall 1 \leq i < J \) and \( N_{kk} = -\frac{\partial \Pi}{\partial q_k} + \frac{\partial \Pi}{\partial q_k} \) \( \forall k \neq j. \) The proof is complete if we can establish that \( \det(N)_{|\mathcal{X} = \mathcal{X}'} > 0 \). Using Theorem 4 in [56] it can be seen that all principal minors of \( N \) are positive if \( N \) is diagonally dominant with positive diagonal entries and negative off diagonal entries. The rest of the proof is devoted to establishing that \( N \) satisfies these properties when \( \mathcal{X} = \mathcal{X}' \).
It follows from Theorem 1, that \( \frac{\partial \Pi}{\partial p} |_{p=0} < 0 \), therefore \( N_P(0, x) = - \frac{\partial \Pi}{\partial p} + \frac{\partial^2 \Pi}{\partial p^2} > 0 \). Using (A.1) and (A.2), the second derivative of \( \Pi_j \) with respect to \( p_k \) and \( p_j \) can be written as

\[
\frac{\partial^2 \Pi_j}{\partial p_k \partial p_j} |_{p=0} = \int_A \frac{\partial D_j(p, x)}{\partial p_k} \cdot f_j(x) dx + \int_A \frac{\partial^2 D_j(p, x)}{\partial p_k \partial p_j} \cdot f_j(x) dx (1 - F_j(z_j)) \quad (B.3)
\]

or

\[
\frac{\partial^2 \Pi_j}{\partial p_k \partial p_j} |_{p=0} = \int_A \frac{\partial D_j(p, x)}{\partial p_k} \cdot f_j(x) dx - \int_A \frac{\partial^2 D_j(p, x)}{\partial p_k \partial p_j} \cdot f_j(x) dx (1 - F_j(z_j)) + \int_A \frac{\partial^2 \Pi_j}{\partial p_k \partial p_j} \cdot f_j(x) dx (1 - F_j(z_j)) + \int_A \frac{\partial D_j(p, x)}{\partial p_k} \cdot f_j(x) dx \quad (B.5)
\]

\[
+ \int_A \frac{\partial^2 \Pi_j}{\partial p_k \partial p_j} \cdot f_j(x) dx (1 - F_j(z_j)) + \int_A \frac{\partial D_j(p, x)}{\partial p_k} \cdot f_j(x) dx \quad (B.6)
\]

To study the sign of \( N_k \), we distinguish two cases.

**First case:** \( \frac{\partial D_j(p, x)}{\partial p} \geq 0 \)

In this case, \( \frac{\partial D_j(p, x)}{\partial p} \geq 0 \), where C1 and C2 are given by the respective terms in (B.4) and (B.3). Because \( \frac{\partial D_j(p, x)}{\partial p} \geq 0 \), term C2 is nonnegative.

If \( D_j(p, x) \) satisfies Assumption 3.ii), \( \frac{\partial D_j(p, x)}{\partial p} \) is decreasing in \( x \)

\[
C_1 = \left( p_j + \lambda_j^{+} - \lambda_j^{-} \right) \int_A \frac{\partial D_j(p, x)}{\partial p} \cdot f_j(x) dx - \lambda_j^{-} \int_A \frac{\partial^2 D_j(p, x)}{\partial p \partial p_j} \cdot f_j(x) dx \geq 0.
\]

If \( D_j(p, x) \) satisfies Assumption 3.iii), the same arguments used in the proof of Theorem 1 will imply that term C1 is non-negative.

**Second case:** \( \frac{\partial D_j(p, x)}{\partial p} \leq 0 \)

According to Assumption 3.iii), \( \frac{\partial D_j(p, x)}{\partial p} \leq 0 \), \( \frac{\partial D_j(p, x)}{\partial p} \) is increasing in \( x \) and \( \frac{\partial^2 D_j(p, x)}{\partial p \partial p_j} \) is independent of \( x \). In this case, we have \( \frac{\partial D_j(p, x)}{\partial p} |_{p=0} = D_1 + D_2 \), where

\[
D_1 \text{ and } D_2 \text{ are given by the respective terms in (B.5) and (B.6). Since } \frac{\partial D_j(p, x)}{\partial p} \text{ is increasing in } x, \text{ we get }
\]

\[
D_1 = \int_A \frac{\partial D_j(p, x)}{\partial p} \cdot f_j(x) dx + \int_A \frac{\partial^2 D_j(p, x)}{\partial p \partial p_j} \cdot f_j(x) dx \left( 1 - F_j(z_j) \right) + \int_A \frac{\partial D_j(p, x)}{\partial p} \cdot f_j(x) dx \left( 1 - F_j(z_j) \right) + \int_A \frac{\partial D_j(p, x)}{\partial p} \cdot f_j(x) dx \left( 1 - F_j(z_j) \right) + \int_A \frac{\partial D_j(p, x)}{\partial p} \cdot f_j(x) dx \left( 1 - F_j(z_j) \right).
\]

The last equality holds because \( \frac{\partial \Pi_j}{\partial p} \geq 0 \) (Assumption 2.iiii) and equation (A.5)).

Now, because \( \frac{\partial D_j(p, x)}{\partial p} \) is independent of \( x \), we obtain

\[
D_2 = \int_A \frac{\partial D_j(p, x)}{\partial p} \cdot f_j(x) dx + \int_A \frac{\partial^2 D_j(p, x)}{\partial p \partial p_j} \cdot f_j(x) dx \left( 1 - F_j(z_j) \right) + \int_A \frac{\partial D_j(p, x)}{\partial p} \cdot f_j(x) dx \left( 1 - F_j(z_j) \right) + \int_A \frac{\partial D_j(p, x)}{\partial p} \cdot f_j(x) dx \left( 1 - F_j(z_j) \right) + \int_A \frac{\partial D_j(p, x)}{\partial p} \cdot f_j(x) dx \left( 1 - F_j(z_j) \right).
\]
Since $\eta_k(p;x)$ is increasing in $x$ (Assumption 2.ii), we have $\frac{\partial \eta_k(p;x)}{\partial x} \geq 0$, which implies that

$$D_2 \geq \frac{\partial \eta_k(p;A)}{\partial \ell_1} D_1(p,A) + \frac{\partial D_1(p,A)}{\partial \ell_1} = 0.$$ 

This implies that $-\frac{\partial \eta_k(p;A)}{\partial \ell_1} \geq 0$ in assumptions 3.ii), 3.iii) and 3.iv). Since $\delta_j > 0, \gamma_j \leq 0$ (Appendix A) and $\frac{\partial \eta_k(p;x)}{\partial x} > 0$, we have $N_{k,x} > 0$.

Now, it only remains to show that $N$ is diagonally dominant. In fact, it can be seen that

$$\left| N_{k,x} - \sum_{k,j} N_{j,x} \right| = \frac{\partial \Pi_j}{\partial \ell_2} \sum_{k,j} \frac{\partial \Pi_j}{\partial \ell_2} + \gamma_j \int \frac{\partial D_1(p,A)}{\partial \ell_2} + \sum_{k,j} \frac{\partial D_1(p,A)}{\partial \ell_2} + \beta_j \right) > \frac{\partial \Pi_j}{\partial \ell_2} - \sum_{k,j} \frac{\partial \Pi_j}{\partial \ell_2} + \gamma_j (1 - F_j(z_j)).$$

The last inequality holds because $\frac{\partial \Pi_j}{\partial \ell_2} + \sum_{k,j} \frac{\partial \Pi_j}{\partial \ell_2} < 0$ and $\frac{\partial \Pi_j}{\partial \ell_2} > 1$. To simplify notations, let $\mathcal{S}_j = -\frac{\partial \eta_k(p;A)}{\partial \ell_1} - \sum_{k,j} \frac{\partial \eta_k(p;x)}{\partial x} + \gamma_j (1 - F_j(z_j))$. From equations (A.8)-(B.6), $\mathcal{S}_j$ can be expressed in two ways:

$$\mathcal{S}_j = \left( p_j + \lambda_j - \lambda_j' \right) \int_0^1 \left( \frac{\partial D(p,x)}{\partial \ell_2} + \sum_{k,j} \frac{\partial D(p,x)}{\partial \ell_2} \right) f_j(x) \mathrm{d}x - \int_0^1 \left( \frac{\partial D(p,x)}{\partial \ell_2} + \sum_{k,j} \frac{\partial D(p,x)}{\partial \ell_2} \right) f_j(x) \mathrm{d}x - \left( \frac{\partial D(p,x)}{\partial \ell_2} + \sum_{k,j} \frac{\partial D(p,x)}{\partial \ell_2} \right) \left( 1 - F_j(z) \right).$$

or

$$\mathcal{S}_j = \int_0^1 \left( \frac{\partial \varepsilon_j(p,x)}{\partial \ell_2} + \sum_{k,j} \frac{\partial \varepsilon_j(p,x)}{\partial \ell_2} \right) \frac{\partial D(p,x)}{\partial \ell_2} \left( 1 - F_j(x) \right) \mathrm{d}x$$

As in the proof of Theorem 1, the rest of the argument for the sign of $\mathcal{S}_j$ is divided into two cases.

**First case:** $\frac{\partial D(p,x)}{\partial \ell_2} + \sum_{k,j} \frac{\partial D(p,x)}{\partial \ell_2} \leq 0$

In this case, $\mathcal{S}_j = E_1 + E_2$, where $E_1$ and $E_2$ are given by the terms in (B.7) and (B.8), respectively. Because $\frac{\partial D(p,x)}{\partial \ell_2} + \sum_{k,j} \frac{\partial D(p,x)}{\partial \ell_2} \leq 0$, term $E_2$ is non-negative. If $D_j(p,x)$ satisfies Assumption 3.ii), $\frac{\partial D(p,x)}{\partial \ell_2} + \sum_{k,j} \frac{\partial D(p,x)}{\partial \ell_2}$ is increasing in $x$ and

$$E_1 \geq - \left( \frac{\partial D(p,x)}{\partial \ell_2} + \sum_{k,j} \frac{\partial D(p,x)}{\partial \ell_2} \right) \left( p_j + \lambda_j - \lambda_j' \right) F_j(z_j) - \lambda_j = \left( \frac{\partial D(p,x)}{\partial \ell_2} + \sum_{k,j} \frac{\partial D(p,x)}{\partial \ell_2} \right) \left( p_j - \lambda_j - p_j \right) \geq 0.$$
Second case: \( \frac{\partial^2 D_i(p,x)}{\partial p_j \partial p_k} + \sum_{k 
eq j} \frac{\partial^2 D_i(p,x)}{\partial p_k \partial p_k} \geq 0. \)

Based on Assumption 3.iv), \( \frac{\partial^2 D_i(p,x)}{\partial p_j \partial p_k} + \sum_{k 
eq j} \frac{\partial^2 D_i(p,x)}{\partial p_k \partial p_k} \leq 0, \quad \frac{\partial^2 D_i(p,x)}{\partial p_j \partial p_k} \) is decreasing in \( x \) and \( \frac{\partial^2 D_i(p,x)}{\partial p_k \partial p_k} \) is independent of \( x \). In this case, \( \mathcal{B}_i = F_1 + F_2 + F_3 \), where \( F_1, F_2, \) and \( F_3 \) are given by the terms in (B.9), (B.10), and (B.11), respectively. Because \( \frac{\partial^2 D_i(p,x)}{\partial p_j \partial p_k} \geq 0 \) term \( F_2 \) is non-negative. Since \( \frac{\partial^2 D_i(p,x)}{\partial p_j \partial p_k} \) is decreasing in \( x \).

Second case:

\[
F_1 \geq \frac{\partial x_j}{\partial x} \sum_{j=1}^{n_x} \left[ \int_{p_j}^{P_j} z_j \, \frac{\partial D_j(p,x)}{\partial p_j} \, (1 - F_j(x)) \, dx - \int_{P_j}^{\bar{p}_j} z_j \, \frac{\partial D_j(p,x)}{\partial p_j} \, (1 - F_j(x)) \, dx \right]
\]

\[
= \frac{\partial x_j}{\partial x} \sum_{j=1}^{n_x} \left[ D_j(p,A_j) + \int_{p_j}^{P_j} \frac{\partial D_j(p,x)}{\partial p_j} \, (1 - F_j(x)) \, dx \right] \geq 0.
\]

The last equality holds because \( \frac{\partial^2 D_j(p,x)}{\partial p_j \partial p_k} + \sum_{k \neq j} \frac{\partial^2 D_j(p,x)}{\partial p_k \partial p_k} \geq 0 \) (Assumption 2.iii) and equation (A.5).

Now, because \( \frac{\partial^2 D_j(p,x)}{\partial p_j \partial p_k} \) is independent of \( x \),

\[
F_3 = \frac{\partial^2 D_i(p,A_i)}{\partial p_j \partial p_k} + \sum_{k 
eq j} \frac{\partial^2 D_i(p,A_i)}{\partial p_k \partial p_k}
\]

\[
= \frac{\partial^2 D_i(p,A_i)}{\partial p_j \partial p_k} + \sum_{k 
eq j} \frac{\partial^2 D_i(p,A_i)}{\partial p_k \partial p_k} \frac{\partial D_i(p,A_i)}{\partial p_k} D_i(p,A_i) - \left( \frac{\partial D_i(p,A_i)}{\partial p_j} + \sum_{k 
eq j} \frac{\partial D_i(p,A_i)}{\partial p_k} \frac{\partial D_i(p,A_i)}{\partial p_k} \right) D_i(p,A_i)
\]

Since \( \frac{\partial^2 D_i(p,A_i)}{\partial p_j \partial p_k} + \sum_{k 
eq j} \frac{\partial^2 D_i(p,A_i)}{\partial p_k \partial p_k} \geq 0 \) (Assumption 2.ii), we have

\[
F_3 \geq \frac{\partial^2 D_i(p,A_i)}{\partial p_j \partial p_k} D_i(p,A_i) - \left( \frac{\partial D_i(p,A_i)}{\partial p_j} + \sum_{k 
eq j} \frac{\partial D_i(p,A_i)}{\partial p_k} \frac{\partial D_i(p,A_i)}{\partial p_k} \right) D_i(p,A_i)
\]

which implies that

\[
F_3 \geq \frac{\partial^2 D_i(p,A_i)}{\partial p_j \partial p_k} D_i(p,A_i) - \left( \frac{\partial D_i(p,A_i)}{\partial p_j} + \sum_{k 
eq j} \frac{\partial D_i(p,A_i)}{\partial p_k} \frac{\partial D_i(p,A_i)}{\partial p_k} \right) D_i(p,A_i) = 0.
\]

Consequently, \( \mathcal{B}_i > 0 \) for assumptions 3.i), 3.ii), 3.iii) and 3.iv) and matrix \( N \) is strictly diagonally dominant with positive diagonal and negative off-diagonal terms, implying that \( \det(N)|_{x=x} > 0. \)

Appendix C. Sensitivity analysis

For simplicity, we examine the special case when \( D_j = D_j(p_j,z_j) \). Using the dominance effect among retailers, the proofs can be easily extended to the general case when \( D_j = D_j(p_j,z_j) \).

By definition of \( z_j, D_j(p_j,z_j) = z_j = \sum_{i=1}^{n_z} q_{i,j} \), with \( z_j = F_j^{-1}\left( \frac{p_j - q_{i,j}}{p_j + q_{i,j} - q_{i,j}} \right) \) and \( \rho_j = \frac{\partial q_{i,j}}{\partial p_j} \). Therefore, \( \frac{\partial q_{i,j}}{\partial q_{i,j}} = \frac{\partial q_{i,j}}{\partial q_{i,j}} = 1, \)
2. ...1. which is equivalent to \( \frac{\partial^2 c_i(q^*_i)}{\partial q^*_i \partial q^*_i} = \frac{\partial^2 c_i(q_i)}{\partial q_i \partial q_i} \). That implies that \( \frac{\partial q^*_i}{\partial q_i} = \frac{\partial q_i}{\partial q_i} \), \( \forall i = 1, 2, \ldots, I - 1 \).

C1. Proof of Proposition 1

Note that at the equilibrium, \( D_j(p^*_j, q^*_j) = s_j^* = \sum_{l=1}^{I} q_{l0}^* \). Taking the derivative with respect to \( c_j \) yields

\[
\frac{\partial D_j}{\partial c_j} = \frac{\partial D_j}{\partial p_j} \left( \frac{\partial p_j}{\partial c_j} \right) + \frac{\partial D_j}{\partial q_j} \left( \frac{\partial q_j}{\partial c_j} \right) - \frac{\partial^2 D_j}{\partial q_j \partial q_j} \left( \frac{\partial q_j}{\partial c_j} \right)^2.
\]

which implies that

\[
\frac{\partial q^*_j}{\partial c_j} \left[ 1 + a_j + \frac{\partial^2 c_i(q_i)}{\partial q_i \partial q_i} q^*_i \right] = \frac{\partial q^*_j}{\partial c_j} - \beta_j, \quad \text{where} \quad a_j = \sum_{l=1}^{I} \frac{\partial^2 c_i(q_i)}{\partial q_i \partial q_i} q_{l0}^* > 0, \quad \beta_j = \frac{\partial q^*_j}{\partial c_j} \left( \frac{\partial^2 c_i(q_i)}{\partial q_i \partial q_i} q_{l0}^* \right) > 0 \quad \text{and} \quad \gamma_j = \frac{\partial q^*_j}{\partial c_j} \left( \frac{\partial^2 c_i(q_i)}{\partial q_i \partial q_i} q_{l0}^* \right) \leq 0 \quad \text{(Appendix A)}. \]

Next, taking the derivative of the equilibrium equation \( dp_j/\partial p_j = 0 \) with respect to \( c_j \) gives

\[
\frac{\partial^2 D_j}{\partial q_j \partial q_j} = 0. \quad \text{Using the above and few algebraic manipulations shows that} \quad \frac{\partial q_j}{\partial c_j} = \frac{\partial^2 c_i(q_i)}{\partial q_i \partial q_i} q_{l0}^*. \quad \text{where} \quad M_j = \frac{\partial^2 c_i(q_i)}{\partial q_i \partial q_i} q_{l0}^* \quad \text{From Appendix A, we know that at equilibrium} \quad \frac{\partial q_j}{\partial p_j} \leq 0, \quad \text{so} \quad M_j \leq 0 \quad \text{and therefore} \quad \frac{\partial q_j}{\partial c_j} \geq 0. \quad \text{To obtain the sign of} \quad \frac{\partial q_j}{\partial c_j} \quad \text{it can be seen using the formula} \quad \frac{\partial q_j}{\partial c_j} \quad \text{that the derivative of} \quad q_j \quad \text{with respect to} \quad c_j \quad \text{simplifies to}
\]

\[
\frac{\partial q_j}{\partial c_j} = \frac{-M_j}{\frac{\partial^2 c_i(q_i)}{\partial q_i \partial q_i} M_j}.
\]

Since \( M_j \leq 0, \frac{\partial q_j}{\partial c_j} \leq 0 \quad \text{and} \quad \frac{\partial q_j}{\partial c_j} \leq 0, \forall i = 1, 2, \ldots, I - 1. \quad \text{Note that at equilibrium,} \quad \sum_{i=1}^{N} q_{i0}^* = \sum_{i=1}^{I} q_{i0}^* \quad \text{and therefore} \quad \frac{\partial q_j}{\partial c_j} \leq 0, \forall n = 1, 2, \ldots, N \quad \text{at the equilibrium and using arguments similar to those in C.1 yields}

C2. Proof of Proposition 2

The arguments used in this subsection are quite similar to those presented in subsection C.1, with the only difference being that derivatives are taken with respect to \( \lambda_j \). In particular, following the same steps, it can be easily shown that

\[
\frac{\partial q_j}{\partial \lambda_j} = \left( \frac{\partial q_j}{\partial c_j} \right) \left( \frac{\partial^2 c_i(q_i)}{\partial q_i \partial q_i} q_{l0}^* \right) \left( \frac{\partial q_j}{\partial c_j} \right) \left( 1 + a_j + \frac{\partial^2 c_i(q_i)}{\partial q_i \partial q_i} q_{l0}^* \right).
\]

Computing \( \frac{\partial q_j}{\partial \lambda_j} \) at the equilibrium and using arguments similar to those in C.1 yields
\[
\frac{dx_j^*}{dx_j} = \frac{\left(1 + \alpha_j\right)\left(1 - F_j\left(z_j^*\right)\right)^{1 - \alpha_j} + \frac{d\alpha_0(q_j)}{dq_j} f_j \int_{z_j}^{\tilde{\lambda}_j} \frac{d\theta}{dq_j} f_j(x) \, dx}{\frac{d\alpha_0(q_j)}{dq_j} (1 + \alpha_j) + \frac{d\alpha_1(q_j)}{dq_j} M_j},
\]
where \(\Delta_j^* = \int_{z_j}^{\tilde{\lambda}_j} \frac{d\theta}{dq_j} f_j(x) \, dx - \frac{d\alpha_0(q_j)}{dq_j} (1 - F_j(z_j^*))\). Substituting in the above formula for \(\frac{dx_j^*}{dx_j}\) gives
\[
\frac{dx_j^*}{dx_j} = \frac{M_j \left(1 - F_j\left(z_j^*\right)\right) + \gamma_j \int_{z_j}^{\tilde{\lambda}_j} \frac{d\theta}{dq_j} f_j(x) \, dx}{\frac{d\alpha_0(q_j)}{dq_j} (1 + \alpha_j) + \frac{d\alpha_1(q_j)}{dq_j} M_j}.
\]

It can be easily verified that \(\frac{dx_j^*}{dx_j} > 0\) for any demand model. The signs of \(\frac{dx_j^*}{dx_j}\) and \(\frac{dx_j^*}{dx_j}\) depend in general on the demand model. In particular, if \(\frac{d\alpha_0(p, x)}{dp}\) is increasing in \(x\), then it is easy to show that \(\frac{d\alpha_0(p, x)}{dp} = A_1 + A_2 \leq 0\) implying that \(\gamma_j \int_{z_j}^{\tilde{\lambda}_j} \frac{d\theta}{dq_j} f_j(x) \, dx + M_j (1 - F_j(z_j^*)) \leq 0\). Therefore, \(\frac{dx_j^*}{dx_j} \geq 0\) but the sign of \(\frac{dx_j^*}{dx_j}\) depends on the model parameters. However, if \(\frac{d\alpha_0(p, x)}{dp}\) is decreasing in \(x\), then \(\Delta_j^* \leq 0\) implying that \(\frac{dx_j^*}{dx_j} > 0\) and the sign of \(\frac{dx_j^*}{dx_j}\) depends on the model parameters. In both cases, \(\frac{dx_j^*}{dx_j} (1 \leq n \leq N, 1 \leq i \leq I)\), \(\frac{dx_j^*}{dx_j}\) and \(\frac{dx_j^*}{dx_j}\) have the same sign as \(\frac{dx_j^*}{dx_j}\).

For the expected profits of raw material suppliers and manufacturers, it can be seen that \(\frac{dx_j^*}{dx_j}\) have the same sign as \(\frac{dx_j^*}{dx_j}\) and \(\frac{dx_j^*}{dx_j} = \sum_{j=1}^t \frac{dx_j^*}{dx_j}\) which has the same sign as \(\frac{dx_j^*}{dx_j}\). On the other hand, \(\frac{dx_j^*}{dx_j} = -\theta_j(p, z_j^*) - \frac{dx_j^*}{dx_j} \sum_{j=1}^t \frac{dx_j^*}{dx_j}\) which depends on the sign of \(\frac{dx_j^*}{dx_j}\). Summing the above yields, \(\frac{dx_j^*}{dx_j} = -\theta_j(p, z_j^*) < 0\), implying that the total profit \(\Pi_j\) decreases with \(\lambda_j\).

**C3. Proof of Proposition 3**

Again, using similar arguments to those in C.1, it can be verified that
\[
\frac{dx_j^*}{dx_j} = \left\{\frac{dp_j^*}{dx_j} f_j + \beta_j F_j\left(z_j^*\right)\right\} \left\{1 + \alpha_j + \frac{d\alpha_0(q_j)}{dq_j} \beta_j\right\},
\]
and using \(\frac{dp_j^*}{dx_j} f_j\), we get
\[
\frac{dp_j^*}{dx_j} = \frac{\left(1 + \alpha_j\right)\left(1 - F_j\left(z_j^*\right)\right)^{1 - \alpha_j} + \frac{d\alpha_0(q_j)}{dq_j} f_j \int_{z_j}^{\tilde{\lambda}_j} \frac{d\theta}{dq_j} f_j(x) \, dx}{\frac{d\alpha_0(q_j)}{dq_j} (1 + \alpha_j) + \frac{d\alpha_1(q_j)}{dq_j} M_j}.
\]

where \(\Delta_j^* = \int_{z_j}^{\tilde{\lambda}_j} \frac{d\theta}{dq_j} f_j(x) \, dx - \frac{d\alpha_0(q_j)}{dq_j} (1 - F_j(z_j^*))\). Substituting in the above formula for \(\frac{dx_j^*}{dx_j}\) gives
\[
\frac{dx_j^*}{dx_j} = \frac{M_j F_j\left(z_j^*\right) + \gamma_j \int_{z_j}^{\tilde{\lambda}_j} \frac{d\theta}{dq_j} f_j(x) \, dx}{\frac{d\alpha_0(q_j)}{dq_j} (1 + \alpha_j) + \frac{d\alpha_1(q_j)}{dq_j} M_j}.
\]

It is also easy to see that \(\frac{dx_j^*}{dx_j} = \frac{d\alpha_0(q_j)}{dq_j} \frac{dx_j^*}{dx_j}\) and that \(\frac{dx_j^*}{dx_j}\) is given by
where

\[
dq_i \lambda_i = \left[ \frac{\partial \Theta_i(z_i)}{\partial p} \left( 1 + a_i \right) + \frac{\partial^2 \Theta_i(z_i)}{\partial q^2} \right]_{\lambda_i} \text{ and } dq_i \alpha_i = \left[ \frac{\partial \Theta_i(z_i)}{\partial \alpha} \right]_{\lambda_i}.
\]

Clearly \( dq_i \lambda_i \geq 0 \) for all models but the signs of \( dq_i \lambda_i \) and \( dq_i \alpha_i \) depend on the demand model. In fact, when \( \frac{\partial \Theta_i(z_i)}{\partial p} \) is increasing in \( x \), then \( \Delta_i^+ \leq 0 \). Therefore, \( dq_i \lambda_i \geq 0 \) and the sign of \( dq_i \alpha_i \) depends on the model parameters. However, when \( \frac{\partial \Theta_i(z_i)}{\partial p} \) is decreasing in \( x \), it is easy to prove that \( \gamma_i \int_{\lambda_i}^{\lambda_i+} \frac{\partial \Theta_i(z_i)}{\partial q} f_i(x)dx + M_i F_i(z_i) \leq 0 \). Therefore, \( dq_i \lambda_i \geq 0 \) and the sign of \( dq_i \alpha_i \) depends on the model parameters. As in the previous subsection, \( \frac{dq_i \lambda_i}{\alpha_i} \) \( (1 \leq i \leq 1) \), \( \frac{dq_i \lambda_i}{\alpha_i} \) and \( \frac{dq_i \alpha_i}{\alpha_i} \) have the same sign as \( dq_i \lambda_i \).

For the expected profits of the retailers, it can be seen that \( \frac{\partial \Pi_i}{\partial q_i} = \lambda_i (p_i^* z_i^*) - \frac{\partial \Theta_i(z_i)}{\partial q} \sum_i q_i \rho_i \), which can be positive or negative depending on the model parameters. Summing all expected profits yields, \( \frac{\partial \Pi}{\partial q} = \lambda_i (p_i^* z_i^*) > 0 \), implying that the total profit \( \Pi \) increases with \( j^+ \).

### C4. Proof of Proposition 4

To proof Proposition 4, we use the mixed linear-exponential model \( D_j(p_j, x) = \mu_j(p_j) + x \theta_j(p_j) \) with \( \mu_j(p) = a_j - b_j p_j \) and examine the effect of varying \( q_j \). Note that the parameter \( q_j \) is used to control the demand level and a change in \( q_j \) results in a change of the demand level without changing demand variability and demand dependence on \( p \). Similar analysis could be carried out for parameters \( b_j \) and \( c_j \). Repeating steps similar to those in the above subsections, we get \( \frac{dq_j}{\alpha_j} = \left( \frac{\partial \Theta_j(z_j)}{\partial p} + 1 \right) \left( 1 + a_j + \frac{\partial^2 \Theta_j(z_j)}{\partial q^2} \right) \). Using the equilibrium equations and adapting the arguments of the previous sections yields

\[
dq_j = \left( 1 + a_j \right) + \left( \gamma_j + \frac{\partial \Theta_j(z_j)}{\partial q} \right) (1 + a_j) + \frac{\partial^2 \Theta_j(z_j)}{\partial q^2} M_j,
\]

and

\[
dq_j = \frac{\partial \Theta_j(z_j)}{\partial \alpha} + \frac{\partial^2 \Theta_j(z_j)}{\partial q^2} M_j.
\]

The term \( \gamma_j + \frac{\partial \Theta_j(z_j)}{\partial q} \) \( \leq 0 \) for the mixed linear-exponential model, implying that \( \frac{dq_j}{\alpha_j} \geq 0 \) and \( \frac{dq_j}{\alpha_j} \geq 0 \). Consequently, \( \frac{dq_j}{\alpha_j} \geq 0 \) and \( \frac{dq_j}{\alpha_j} \geq 0 \). \( \forall j = 1, 2, \ldots I - 1 \), and \( \frac{dq_j}{\alpha_j} > 0 \).

For the total expected profit, it can be seen that \( \frac{\partial \Pi}{\partial q} = (p_j^* + \lambda_j - \lambda_j^*) F_j(z_j^*) - \lambda_j > 0 \) implying that the total profit \( \Pi \) increases with \( q_j \). The expected profits of raw material suppliers and manufacturers increase with \( a_j \) since \( \frac{dq_j}{\alpha_j} > 0 \) and \( \frac{dq_j}{\alpha_j} > 0 \).

### C5. Proof of Proposition 5

To proof Proposition 5, we use the linear demand function \( D_j(p_j, x) = a_j - b_j p_j + m_j x \) and study the effect of varying \( m_j \). Note that for this model, a change in \( m_j \) results in a change of demand variability without changing demand level or demand dependence on \( p \). Mimicking the same steps as in the previous subsections yields

\[
dp_j = \Theta_j(z_j) (1 + a_j) + \left( \gamma_j + \Theta_j(z_j) \right) (1 + a_j) \frac{\partial \Theta_j(z_j)}{\partial q} M_j,
\]

and

\[
dq_j = \frac{\partial \Theta_j(z_j)}{\partial \alpha} + \frac{\partial^2 \Theta_j(z_j)}{\partial q^2} M_j.
\]

where \( \Theta_j(z_j) = \int_{\lambda_j}^{\lambda_j+} (x - z_j) f_j(x)dx = - \int_{\lambda_j}^{\lambda_j+} x f_j(x)dx - z_j (1 - F_j(z_j)) \).

The signs of \( \frac{dp_j}{\alpha_j} \) and \( \frac{dq_j}{\alpha_j} \) depend on the sign of \( z_j \). In fact, if \( z_j \leq 0 \), then \( \Theta_j(z_j) (1 + a_j) + \left( \gamma_j + \Theta_j(z_j) \right) \frac{\partial \Theta_j(z_j)}{\partial q} \geq 0 \), implying that \( \frac{dp_j}{\alpha_j} \leq 0 \). Calculations
show that the sign of $\frac{d\Pi}{dm}$ depends on the model parameters. If $z_j^* \geq 0$, then $x^*p_j + \gamma\Theta(z_j^*) \leq 0$, implying that $\frac{d\Pi}{dm} = 0$. For $\frac{dx}{dm}$, its sign depends on the model parameters.

For the expected profits, straightforward computation shows that $\frac{d\Pi}{dm} = (p_j^* + \lambda^2) \int_0^\infty f_j(x) < 0$ implying that the total profit II decreases with $m$. For the expected profits of raw material suppliers, manufacturers, and retailers, it can be seen that $\frac{d\Pi}{dq}$ and $\frac{dx}{dq}$ have the same sign as $\frac{d\Pi}{dm}$ and the sign of $\frac{dx}{dm}$ depends on the model parameters.

References


