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Computation of a well-conditioned dynamic stiffness matrix for elastic layers overlying a half-space

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Abstract. In the context of range–independent solid media, we propose a well–conditioned dynamic stiffness matrix for an elastic layer sitting over an elastic half–space. This formulation overcomes the well–known problem of numerical ill-conditioning when solving the system of equations for deep-layered strata. The methodology involves the exact solutions of transformed ordinary differential equations in the wavenumber domain, a projection method based on the transformed equations with respect to the depth coordinate. By re–arranging the transformed equations the solutions remain numerically well-conditioned for all layer depths. The inverse transforms are achieved with a numerical quadrature method and the results presented include actual displacement fields in the near-field of the load.

1. Introduction

A two–dimensional model is considered to demonstrate the effect of a harmonic finite strip–load over layered strata, [1]. The results derived by Fourier transform are valid for any frequency and more importantly any depth of layer. In principle, following the well-known traditional methods, [2], we could use displacement and stress–continuity boundary conditions at the bottom of the layer with equations at the ground surface to generate equations for four subsequent unknowns of stress and displacement. However, this direct approach leads to formidable numerical problems. The reason for this work is that if traditional expressions for the characteristic wave functions, such as cosh or sinh, are employed these can have a dramatic effect on the numerical evaluation of solutions. Problems arise due to the cancellation or division of either very small or very large numbers. To overcome this Karasalo [3] derived a well–conditioned propagator matrix for radially symmetric problems. In this work, though, we construct a single stiffness matrix for the physical layer for plane–strain problems which conveniently avoids these difficulties. We therefore deduce a new global dynamic stiffness matrix for functions that do not cause numerical problems. A *scaled* dynamic stiffness matrix for the layer is derived. To do this the vibration components in the wavenumber domain for layer depths and a half–space are considered and arranged into a single matrix formulation. For the forced response, the load is modelled as an infinite strip, so that the problem is plane. The purpose of the present study is to present a computational method which does not suffer numerical evaluation difficulties when predicting vibration transmission, in particular its attenuation on the surface of a deep layer. The usefulness of the method is illustrated by presenting numerical results from two potentially computationally intensive application examples.
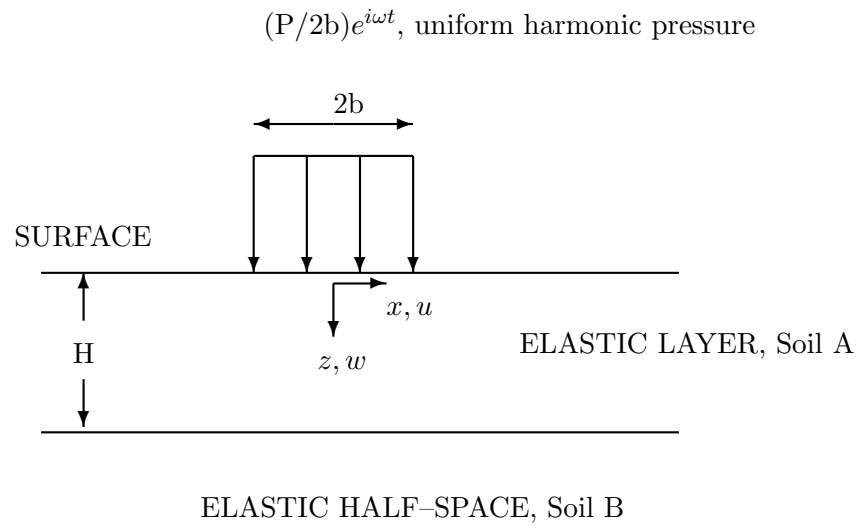
$(\text{P}/2\text{b})e^{i\omega t}$, uniform harmonic pressure

2b

SURFACE

x, u

H

z, w

ELASTIC LAYER, Soil A

ELASTIC HALF–SPACE, Soil B

Figure 1. Diagram of the model

Although not developed in this short–communication, solutions to characteristic equations which establish wave propagation parameters can now be determined more efficiently for dynamic [4] or moving–load problems [5], for example, due to the reduced number of equations.

2. Vibration transmission

The model considered is shown in Figure 1. The strip load has a length 2*b*, and is aligned with respect to the *z*-axis. It rests on an homogeneous, isotropic, elastic layer, with material properties E (Young's modulus), ρ (density) and ν (Poisson's ratio). An harmonic vertical load acts uniformly over the strip. The elastic layer of finite depth, H, of homogeneous and isotropic material, overlies a half–space of flexible material. The model is two–dimensional, and the co–ordinate system and parameters are shown in Figure 1.

Much of the analysis necessary for the derivation of the dynamic stiffness matrix has been presented in references [1], [2] and [6], so this will only be briefly summarized. For plane strains the behaviour of the elastic material is described by Navier's elastodynamic equations, [1]. In the absence of body forces, these can be written as:

$$(\lambda + \mu)\frac{\partial \Delta}{\partial x} \; + \; \mu \nabla^2 u \; = \; \rho\frac{\partial^2 u}{\partial t^2}, \;\; (\lambda + \mu)\frac{\partial \Delta}{\partial z} \; + \; \mu \nabla^2 w \; = \; \rho\frac{\partial^2 w}{\partial t^2}, \tag{1}$$

where u, w are the components of the displacement in the x and z directions and Δ is the dilatation, ρ is the density of the material and λ, μ are complex Lamé constants. The stress–strain relations are also included here :

$$\sigma_{zx} \; = \; \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \sigma_{zz} \; = \; \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) \; + \; 2\mu\frac{\partial w}{\partial z}, \tag{2}$$

The quantities c_1 and c_2 are respectively the P and S wave speeds, given by:

$$c_1^2 = \frac{\lambda + 2\mu}{\rho} = \frac{E(1-\nu)}{\rho(1+\nu)(1-2\nu)}, \;\; c_2^2 = \frac{\mu}{\rho} = \frac{E}{2\rho(1+\nu)}. \tag{3}$$

Table 1. Material properties

Soil	A	B
$E\ (Pa)$	2.69×10^8	204×10^8
$\rho\ (kgm^{-3})$	1550	2450
ν	0.257	0.179
$c_R\ (ms^{-1})$	242	1706
$c_2\ (ms^{-1})$	263	1879
$c_1\ (ms^{-1})$	459	3005

the boundary conditions for this problem are a uniform strip–load if width $2b$ acting over the surface and at $z = H$ continuity of displacement and traction between the upper and lower–media. Equation (1) may be solved by introducing potentials ϕ and H such that

$$u = \partial\phi/\partial x - \partial H/\partial z \text{ and } w = \partial\phi/\partial z + \partial H/\partial x \tag{4}$$

where, given wavenumbers $k_1 = \omega/c_1$ and $k_2 = \omega/c_2$

$$\alpha_1^2 = \zeta^2 - k_1^2 \text{ and } \alpha_2^2 = \zeta^2 - k_2^2. \tag{5}$$

It is common practice to write solutions to the ensuing ordinary differential equations in terms of cosh and sinh functions. For computational purposes this choice of characteristic functions is not convenient for problems involving spatial domains chosen to be extremely deep. Hence, we propose the general solutions may be written as a *scaled formulation*:

$$\overline{\phi} = A_1\, e^{-\alpha_1 z} + A_2\, e^{\alpha_1(z-H)}, 0 < z < H$$

$$\tag{6}$$

$$\overline{H} = B_1\, e^{-\alpha_2 z} + B_2\, e^{\alpha_2(z-H)},\ 0 < z < H.$$

The reasons for choosing the scaled exponential characteristic functions over the hyperbolic functions is clear. Essentially, this choice ensures the characteristic functions do not grow unbounded with depth, H including other expressions involving variables, α_1 or α_2, Eq. (5).

Substituting the values $z = 0$ and $z = H$ into the equations (6) yields the first matrix equation

$$\{\overline{u}\} = [C]\{A\}. \tag{7}$$

where $\overline{u} = [i\overline{w}_0,\ \overline{u}_0,\ i\overline{w}_H,\ \overline{u}_H]^T$ and the 4x4 complex-valued matrix $[C]$ is given by

$$[C] = \begin{bmatrix} -i\alpha_1 & i\alpha_1 e^{-\alpha_1 H} & -\zeta & -\zeta e^{-\alpha_2 H} \\ i\zeta & i\zeta e^{-\alpha_1 H} & \alpha_2 & -\alpha_2 e^{-\alpha_2 H} \\ -i\alpha_1 e^{-\alpha_1 H} & i\alpha_1 & -\zeta e^{-\alpha_2 H} & -\zeta \\ i\zeta e^{-\alpha_1 H} & i\zeta & \alpha_2 e^{-\alpha_2 H} & -\alpha_2 \end{bmatrix}. \tag{8}$$

Now, further developing the system of equations from the stress equations (2),

$$\{\overline{\sigma}\} = [S]\{A\}, \tag{9}$$

$$[S] = \begin{bmatrix} -i\alpha_1^2(\lambda+2\mu)+i\lambda\zeta^2 & \left(-i\alpha_1^2(\lambda+2\mu)+i\lambda\zeta^2\right)g_1 & -2\mu\zeta\alpha_2 & 2\mu\zeta\alpha_2\zeta\,g_2 \\ 2i\mu\zeta\alpha_1 & -2i\mu\zeta\alpha_1\zeta\,g_1 & \mu\left(\alpha_2^2+\zeta^2\right) & \mu\left(\alpha_2^2+\zeta^2\right)g_2 \\ i\left(\alpha_1^2(\lambda+2\mu)-\lambda\zeta^2\right)g_1 & i\left(\alpha_1^2(\lambda+2\mu)-\lambda\zeta^2\right) & 2\mu\zeta\alpha_2\,g_2 & -2\mu\zeta\alpha_2 \\ -2i\mu\zeta\alpha_1\,g_1 & 2i\mu\zeta\alpha_1 & -\mu\left(\alpha_2^2+\zeta^2\right)g_2 & -\mu\left(\alpha_2^2+\zeta^2\right) \end{bmatrix} \tag{10}$$

and $g_i = e^{-\alpha_i H}, i = 1, 2$ and $\overline{\sigma} = [-i\overline{\sigma}_0, -\overline{\tau}_0, i\overline{\sigma}_H, \overline{\tau}_H]^T$. We now combine equations (7) and (9), to arrive at a single matrix expression which expresses the displacements and stresses at the surface and the interface in the wavenumber–domain:

$$[T]\{\overline{u}\} = \{\overline{\sigma}\}. \qquad (11)$$

where $[T] = [S][C]^{-1}$. The algebraically complicated unstructured matrix $[T]$ is given in the Appendix. Specifically $[T]$ is the dynamic stiffness matrix for a single elastic layer valid for any depth $H > 0$. To include the half–space, we utilize the matrix equations presented in [6] which leads to a matrix system

$$[P] = \frac{1}{D} \begin{bmatrix} (\lambda + 2\mu)\alpha_2 k_1^2 & 2\mu\zeta(\alpha_1\alpha_2 - \zeta^2) + (\lambda + 2\mu)\zeta k_1^2 \\ 2\mu\zeta(\alpha_1\alpha_2 - \zeta^2) + \mu\zeta k_2^2 & \mu\alpha_1 k_2^2 \end{bmatrix} \begin{bmatrix} i\overline{w}_H \\ \overline{u}_H \end{bmatrix} = \begin{bmatrix} i\overline{\sigma}_H \\ \overline{\tau}_H \end{bmatrix} (12)$$

where $D = 1/(\alpha_1\alpha_2 - \zeta^2)$ and it is understood the soil parameters are related to the half–space below the upper–layer. Equations can now be combined to give a single matrix equation for an elastic layer over an elastic halfspace, involving the *scaled* stiffness matrix for the elastic layer $[T_{ij}]$ and the matrix for the halfspace $[P_{ij}]$. The general matrix form for any global domain becomes a 4x4 complex valued matrix

$$[T]_G = \begin{bmatrix} T & T & T & T \\ T & T & T & T \\ T & T & T\text{-}P & T\text{-}P \\ T & T & T\text{-}P & T\text{-}P \end{bmatrix} \qquad (13)$$

It is straightforward to generalise this technique to n elastic layers over a half–space where the size of the dynamic stiffness matrix will become a single complex–valued $2(n + 1)$ x $2(n + 1)$ matrix.

3. Problems in numerical evaluation of stiffness matrix

Generally, for non–dimensional wavenumbers $k_s h \geq 13$, where k_s is the shear wavenumber, the conventional approach "breaks down". That is, for depths greater than around two shear wavelengths, $h \geq 2\lambda_s$ a numerical bottleneck problem arises when solving the linear system of algebraic equations. Equally for high frequency computations can become ill–conditioned. Note that we cannot show results where "bottleneck" occurs or is about to occur as matrix elements can become unbounded so solutions are not presentable.

The choice of projected method permits a stable numerical evaluation for entries in the stiffness matrix for all soil types, frequencies and layer–depths. This avoids numerical round–off errors especially division of large numbers by small numbers and the subtraction of very large numbers. For example, due to round–off errors, it can be shown that for large $zeta_1, \zeta_2 > 0$ the relative error corresponding to the evaluation of the subtraction of two functions, such as $|\cosh(\zeta_1) - \cosh(\zeta_2)|$ can grow like $\exp(|\zeta|)$, $\zeta_1 < \zeta < \zeta_2$ where the difference $|\zeta_1 - \zeta_2|$ is small. In themselves the numerical evaluation of hyperbolic functions cosh and sinh can also be problematic for large values of their arguments.

4. Numerical results

Soil characteristics values used are presented in Table 1, that is two sets of soil characteristics which represent different conditions for a single frequency, 64 Hz on a strip of width $2b = 1.5$ m subjected to a uniform unit load $P = 1$ N.

Not shown here but functions in wavenumber domain which determine the vibration response for elastic layers over half–space with the same material have been computed. The absolute errors between various cases from shallow to very deep layers were all negligible. Nonetheless the results presented, calculated using the Clenshaw–Curtis numerical quadrature method, are related to the ground parameters shown in Table 1.

5. Free vibration

In Figure 2 is shown the variation of wavenumber with frequency for the first six natural modes. Included in the figure shows the variation of wavenumber for a thick layer of material, soil B, which overlies the half–space of softer material, soil A. Figures 3(a) and (b) compare results
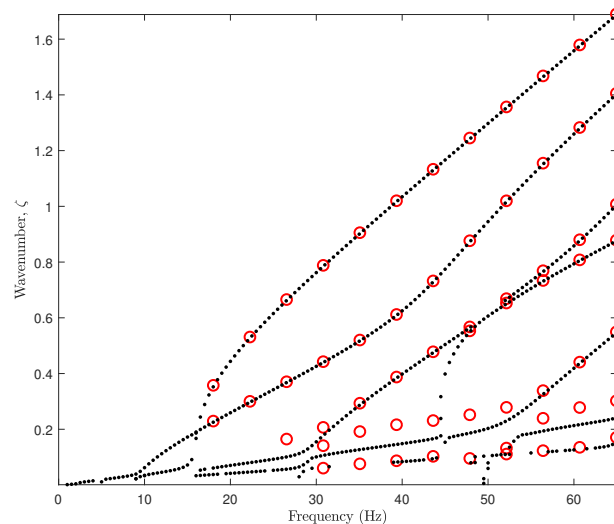

Figure 2. Variation of the wavenumber with frequency of the first six propagating modes for the 7.0m layer over a half–space (dots) and the same layer over thick 9.0m hard strata.

from a simple half–space model [6], to the scaled layered models for 2.0m, 10.0m and 100.0m layers over half–spaces. Especially comparing results between the elastic layer of depth 100.0m compared to the half–space model it is clear that the scaled model approach allows analysis for very deep layers.

6. Conclusions

A two–dimensional model has been developed for investigating the propagation of surface vibration over arbitrary depth elastic–layers. The model consists of an elastic, isotropic and homogeneous layer which overlies a half–space. A well-conditioned dynamic stiffness matrix has been developed for this model, which is derived by projecting the characteristic functions onto the end–points in the depth dimension.

Given the general validity of this formulation for dynamic stiffness matrices many new problems may be modelled, where plane–strain conditions apply, are within easy reach. This method may also be easily developed to take into account sub–layers of different material within the strata and extended to three-dimensional problems..

6.1. Acknowledgements

7. References

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[2] Jones D V and Petyt M 1993 *Journal of Sound and Vibration* **161** 1–18
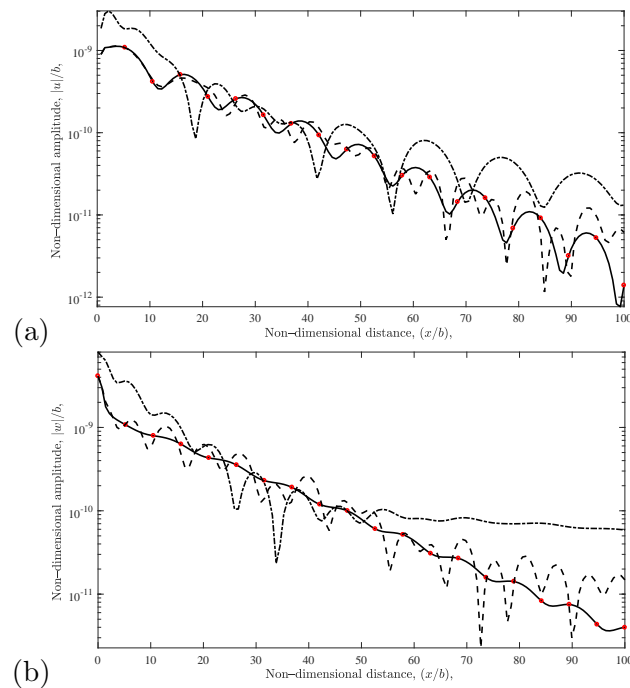[3] Karasalo I 1994 *Journal of Sound and Vibration* **172** 671–688

Figure 3. Absolute values of transformed components horizontal and vertical displacement, plotted against non–dimensional distance, elastic layer (soil A_1) overlying a half–space model (soil B_1), $H = 2.0m$ (dash–dot) and $H = 10.0m$ (dashed) $H = 100.0m$ (solid) at frequency $64Hz$. Red-dots represent displacements calculated using the half–space model, [6].

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Appendix

We follow the notation that the element in the ith row and jth column is denoted T_{ij}. The matrix $[T]$ is however not symmetric but we have $T_{31} = T_{13}$, $T_{32} = -T_{14}$, $T_{33} = T_{11}$, $T_{34} = -T_{12}$, $T_{41} = -T_{23}$, $T_{42} = T_{24}$, $T_{43} = -T_{21}$ and $T_{44} = T_{22}$. The remaining elements of the matrices are as follows:

$$
\begin{aligned}
T_{11} &= -\alpha_2 \left(\lambda + 2\mu\right) k_1^2 \left(\left(\zeta^2 - \alpha_1 \alpha_2\right)^2 \left(e^{-2(\alpha_1+\alpha_2)H} - 1\right) + \right. \\
&\quad \left. \left(\zeta^2 + \alpha_1 \alpha_2\right)^2 \left(e^{-2\alpha_1 H} + e^{-2\alpha_2 H}\right) \right) / D, \tag{14}
\end{aligned}
$$

$$
\begin{aligned}
T_{12} &= \zeta \left\{ \left(\zeta^2 - \alpha_1 \alpha_2\right) \left((\lambda + 2\mu)\alpha_1^2 - 2\mu\alpha_1\alpha_2 - \lambda\zeta^2\right) \left(e^{-(\alpha_1+\alpha_2)2H} + 1\right) + \right. \\
&\quad 4\alpha_1\alpha_2 \left((\lambda + 2\mu)\alpha_1^2 - (\lambda - 2\mu)\zeta^2\right) e^{-(\alpha_1+\alpha_2)H} - \\
&\quad \left. \left(\zeta^2 + \alpha_1\alpha_2\right) \left((\lambda + 2\mu)\alpha_1^2 + 2\alpha_1\alpha_2 - \lambda\zeta^2\right) \left(e^{-2\alpha_1 2H} + e^{-2\alpha_2 H}\right) \right\} / D, \tag{15}
\end{aligned}
$$

$$
\begin{aligned}
T_{21} &= -\mu\zeta \left\{ \left(\zeta^2 - \alpha_1\alpha_2\right)\left(\zeta^2 + \alpha_2^2 - 2\alpha_1\alpha_2\right)\left(e^{-2(\alpha_1+\alpha_2)H} + 1\right) + \right. \\
&\quad 4\alpha_1\alpha_2 \left(\alpha_2^2 + 3\zeta^2\right) e^{-(\alpha_1+\alpha_2)H} + \\
&\quad \left. \left(\zeta^2 + \alpha_1\alpha_2\right)\left(\zeta^2 + 2\alpha_1\alpha_2 + \alpha_2^2\right)\left(e^{-2\alpha_1 H} + e^{-2\alpha_2 H}\right) \right\} / D, \tag{16}
\end{aligned}
$$

$$
T_{22} = \mu\alpha_1 k_2^2 \left\{ \left(\zeta^2 + \alpha_1\alpha_2\right)\left(e^{-2\alpha_1 H} - e^{-2\alpha_2 H}\right) - \left(\zeta^2 - \alpha_1\alpha_2\right)\left(e^{-2(\alpha_1+\alpha_2)H} - 1\right) \right\} / D, \tag{17}
$$

$$
T_{13} = 2\alpha_2 k_1^2 \left(\lambda + 2\mu\right) \left\{ \zeta^2 e^{-\alpha_2 H}\left(e^{-2\alpha_1 H} - 1\right) + \alpha_1\alpha_2 e^{-\alpha_1 H}\left(1 - e^{-2\alpha_2 H}\right) \right\} / D, \tag{18}
$$

$$
T_{14} = -2\alpha_1\alpha_2\zeta k_1^2 \left(\lambda + 2\mu\right) \left\{ e^{-\alpha_1 H}\left(e^{-2\alpha_2 H} + 1\right) - e^{-\alpha_2 H}\left(e^{-2\alpha_1 H} + 1\right) \right\} / D, \tag{19}
$$

$$
T_{23} = 2\mu\zeta\alpha_1\alpha_2 k_2^2 \left\{ e^{-\alpha_1 H}\left(e^{-2\alpha_2 H} + 1\right) - e^{-\alpha_2 H}\left(e^{-2\alpha_1 H} + 1\right) \right\} / D, \tag{20}
$$

$$
T_{24} = -2\mu\alpha_1 k_2^2 \left\{ \zeta^2 e^{-\alpha_1 H}\left(1 - e^{-2\alpha_2 H}\right) - \alpha_1\alpha_2 e^{-\alpha_2 H}\left(1 - e^{-2\alpha_1 H}\right) \right\} / D. \tag{21}
$$

Here

$$
\begin{aligned}
D &= \left(\zeta^2 - \alpha_1\alpha_2\right)^2 \left(\exp(-2(\alpha_1 + \alpha_2)H) + 1\right) - \\
&\quad \left(\zeta^2 + \alpha_1\alpha_2\right)^2 \left(\exp(-2\alpha_1 H) + \exp(-2\alpha_2 H)\right) + \\
&\quad 8\zeta^2\alpha_1\alpha_2 \exp(-(\alpha_1 + \alpha_2)H) \tag{22}
\end{aligned}
$$