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Fractional Bernstein operational matrices for solving integro-differential equations involved by Caputo fractional derivative

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ABSTRACT

The present work is devoted to developing two numerical techniques based on fractional Bernstein polynomials, namely fractional Bernstein operational matrix method, to numerically solving a class of fractional integro-differential equations (FIDEs). The first scheme is introduced based on the idea of operational matrices generated using integration, whereas the second one is based on operational matrices of differentiation using the collocation technique. We apply the Riemann–Liouville and fractional derivative in Caputo's sense on Bernstein polynomials, to obtain the approximate solutions of the proposed FIDEs. We also provide the residual correction procedure for both methods to estimate the absolute errors. Some results of the perturbation and stability analysis of the methods are theoretically and practically presented. We demonstrate the applicability and accuracy of the proposed schemes by a series of numerical examples. The numerical simulation exactly meets the exact solution and reaches almost zero absolute error whenever the exact solution is a polynomial.

We compare the algorithms with some known analytic and semi-analytic methods. As a result, our algorithm based on the Bernstein series solution methods yield better results and show outstanding and optimal performance with high accuracy orders compared with those obtained from the optimal homotopy asymptotic method, standard and perturbed least squares method, CAS and Legendre wavelets method, and fractional Euler wavelet method.

1. Introduction

Fractional integro-differential equations (FIDEs) have attracted a great importance for modeling several problems in fluid mechanics [1], physical system [2] , dynamical system [3], radio astronomy [4], seismology [5], and electron emission [6]. Note that, the analytic solutions are usually hard to obtain. Alternatively, and due to the wide range of applications of FIDEs, the research community has shown a great contribution for developing numerical methods to find approximate solution to these FIDEs. For instance, Ahmed and Elzaki [7] applied some numerical methods to find the comparative study of fractional integral–differential equations. The Least squares method and shifted Chebyshev polynomials are introduced in [8] for solving several types of FIDEs. A numerical technique based on the discrete

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collocation method have been used in [9] to solve a class of FIDEs, hat basis functions [10] and Galerkin method [11]. Recently, the authors in [12] have used Riesz wavelets in $L_2(\mathbb{R})$ to solve singular fractional integro-differential equations with some biological applications. Fractional nonlinear Volterra–Fredholm integral equations involving Atangana–Baleanu fractional derivative are also investigated in [13], and Bernoulli wavelet method for numerical solution of anomalous infiltration in [14].

Bernstein polynomials method (B-polynomials) is one of the most important numerical methods that been considered to solve different type of differential equations. For example, the authors in [15] have employed the B-polynomials to solve a class of ODE system. It is also used to solve third order ODEs with application to fluid flow in [16]. An improvement on the constant in Videnskil's inequality for B-polynomials is introduced in [17]. Fractional Bernstein series solution based on the B-polynomials are introduced to solve diffusion equations with error estimate [18]. See [19–22], and references therein for more details.

FIDEs are widely used in many areas of applications in physics, engineering and many other applied sciences, see for example [23]. It is worth to state that an extensive development of fractional calculus has been concluded to describe many applications that using FIDEs. Here, we consider a class of FIDEs defined by,

$$D^{\alpha}u(x) = u(x) + \lambda_1 \int_0^x \frac{u(t)}{\sqrt{x-t}}dt + \lambda_2 \int_0^x K(x,t)u(t)dt + h(x) \tag{1}$$

under the initial condition,

$$D^{\alpha}u(\delta) = u_i. \quad n-1 < \alpha \le n, \quad n \in N, \quad 0 \le \delta \le R, \tag{2}$$

where $K(x,t)$ and $h(x)$ are known continuous functions on $[0,R]$, $D^{\alpha}u(x)$ denotes the fractional order derivative of the unknown $u(x)$, and δ, λ_i and u_i are constants.

In the present work, two numerical techniques are introduced to solve Equation (1) by the means of the matrix relations between the Bernstein polynomials $B_n(x)$ and their integrations and derivatives. Riemann–Liouville fractional integral operator is applied to introduce Bernstein operational matrix of integration $\Psi(x)$, as well as Caputo sense is applied to introduce Bernstein operational matrix of derivative $\boldsymbol{\Omega}(x)$. To show the efficiency of the proposed methods, we obtain the operational matrices of differentiation for the problem which has non-smooth exact solution. We compare the results with some known methods such as the optimal homotopy asymptotic method, Standard and Perturbed Least Squares Method, CAS Method and Legendre wavelets method and fractional-order Euler functions method.

The structure of this paper is organized as follows. In Section 2, we provide some preliminaries needed on fractional calculus and necessary definitions related to Bernstein polynomials. In Section 3, the main section, we discuss two numerical algorithms, the Bernstein series solution (BSS) by using operational matrices of integration (we call it the BSSI) and the BSS by involving operational matrices of differentiation (we call it the BSSD), to solve the proposed FIDEs. Employing the conditions and applying the Gauss elimination procedure yield the unknown coefficient matrices. To estimate the absolute error and the stability of the methods, we give the residual correction procedure in Section 3.3. Section 4 illustrates some numerical experiments to validate the method for different values of n and the conclusion can be seen in Section 5.

2. Fractional Bernstein polynomials

In this section, we present the definition of Bernstein polynomials and some preliminaries and properties of the fractional calculus needed through out the paper [18,24].

Definition 2.1. A real function $u(x)$, $x > 0$, is said to be in the space C_{μ}, $\mu \in \Re$, if there exists a real number $p > \mu$, such that $u(x) = x^p u_1(x)$, where $u_1(x) \in C(0,\infty)$, and it is said to be in the space C_{μ}^n if and only if $h^{(n)} \in C_{\mu}$, $n \in N$.

Definition 2.2. The Riemann–Liouville fractional integral operator (J^{α}) of order $\alpha \ge 0$, of a function $u \in C_{\mu}$, $\mu \ge -1$, is defined as

$$J^{\alpha}u(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1}u(x)ds \quad (\alpha > 0),$$
$$J^0 u(x) = u(x), \tag{3}$$

where $\Gamma(\alpha)$ is well-known gamma function. Some of the properties of the operator J^{α}, which we will need here, are as follows: For $u \in C_{\mu}$, $\mu \ge -1$, $\alpha, \beta \ge 0$ and $\gamma \ge -1$:

1. $J^{\alpha}J^{\beta}u(x) = J^{\alpha+\beta}u(x)$,
2. $J^{\alpha}J^{\beta}u(x) = J^{\beta}J^{\alpha}u(x)$,
3. $J^{\alpha}x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}x^{\alpha+\gamma}$.

Definition 2.3. The fractional derivative (D^{α}) of $u(t)$, in Caputo's sense is defined as

$$D^{\alpha}u(x)=\frac{1}{\Gamma(n-\alpha)}\int_0^t (x-s)^{n-\alpha-1}u^{(n)}(x)ds, \tag{4}$$

for $n-1<\alpha<n, n\in N, x>0, u\in C_{-1}^{n}$.

The following are two basic properties of the Caputo fractional derivative [25]:

1. Let $u\in C_{-1}^{n}, n\in N$. Than $D^{\alpha}u, 0\le\alpha\le n$ is well defined and $D^{\alpha}u\in C_{-1}$.
2. Let $n-1\le\alpha\le n, n\in N$ and $u\in C_{\mu}^{n}, \mu\ge -1$. Then

$$(J^{\alpha}D^{\alpha})u(x)=u(x)-\sum_{k=0}^{n-1}u^{(k)}(0^{+})\frac{x^k}{k!}. \tag{5}$$

For the Caputo derivative we have

$$D_*^{\alpha}c=0, \quad (c \quad \text{constant}), \tag{6}$$

$$D_*^{\alpha}x^{\beta}=\begin{cases} 0, \quad \text{for} \quad \beta\in N_0 \quad \text{and} \quad \beta<\lceil\alpha\rceil, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}x^{\beta-\alpha}, \quad \text{for} \quad \beta\in N_0 \quad \text{and} \quad \beta\ge\lceil\alpha\rceil \quad \text{or } \beta>\lfloor\alpha\rfloor. \end{cases} \tag{7}$$

Bernstein polynomials of nth degree are given by the following:

$$B_{t,n}(x)=\binom{n}{t}\frac{x^t(R-x)^{n-t}}{R^n}, \quad t=0,1,2,\dots,n \quad x\in[0,R]. \tag{8}$$

By substituting $x\to x^{\alpha}$ into $B_{k,n}(x)$, we obtain $B_{k,n}^{\alpha}(x)$, namely fractional Bernstein polynomials, as

$$B_{t,n}^{\alpha}(x)=\binom{n}{t}\frac{x^{t\alpha}(R-x^{\alpha})^{n-t}}{R^n}, \quad 0<\alpha<1, \\ t=0,1,2,\dots,n \quad x\in[0,R]. \tag{9}$$

3. The numerical techniques

In this section, we provide two solution methods based on the fractional Bernstein polynomials to solve FIDEs numerically. First, we introduce the first method constituted by Bernstein polynomials and operational matrix of integration. The second method is performed based on Bernstein polynomials and operational matrix of differentiation.

3.1. BSSI method

We want to approximate the exact solution by the truncated Bernstein series of degree n. Based on the definition of Bernstein polynomials in Section 2, we can write the approximate solution, Bernstein series solution obtained by operational matrix of integration (BSSI), of Eq. (1) as

$$u(x)=\mathbf{B}_n(x)\mathbf{Z}, \tag{10}$$

where,

$$\mathbf{B}_n(x)=\begin{bmatrix}B_{0,n}(x) & B_{1,n}(x) & B_{2,n}(x) & \cdots & B_{n,n}(x)\end{bmatrix}, \quad \mathbf{Z}=\begin{bmatrix}z_0\\ z_1\\ \vdots\\ z_n\end{bmatrix}.$$

Note that, $[\mathbf{B}_n(x)]^{\mathbf{T}}$ can be written as

$$[\mathbf{B}_n(x)]^{\mathbf{T}}=\begin{bmatrix}B_{0,n}(x)\\ B_{1,n}(x)\\ B_{2,n}(x)\\ \vdots\\ B_{n,n}(x)\end{bmatrix}=\mathbf{X}(x)\mathbf{D}^{\mathbf{T}},$$

where

$$\mathbf{D} = \begin{pmatrix} d_{00} & d_{01} & d_{02} & \dots & d_{0n} \\ d_{10} & d_{11} & d_{12} & \dots & d_{1n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{n0} & d_{n1} & d_{n2} & \dots & d_{nn} \end{pmatrix}, \quad \mathbf{X}(x) = [1 \quad x \quad x^2 \dots x^n],$$

$$d_{ij} = \begin{cases} \frac{(-1)^{j-i}}{R^j}\binom{n}{i}\binom{n-i}{j-i}, & i \le j \\ 0 \quad , & i > j \end{cases}.$$

Now, the approximate solution in (10) will be given as

$$u(x) = \mathbf{X}(x)\mathbf{D}^{\mathbf{T}}\mathbf{Z}. \tag{11}$$

Using Riemann–Liouville fractional integral operator, the relation between the matrix $\mathbf{X}(x)$ and its integration $J^{\alpha}[\mathbf{X}(x)]$ can be introduced as

$$J^{\alpha}[\mathbf{X}(x)] = \left[\frac{\Gamma(1)}{\Gamma(\alpha+1)}x^{\alpha} \quad \frac{\Gamma(2)}{\Gamma(2+\alpha)}x^{1+\alpha} \quad \dots \quad \frac{\Gamma(n+1)}{\Gamma(n+1+\alpha)}x^{n+\alpha} \right]. \tag{12}$$

Therefore, Eq. (12) will be given by the following

$$J^{\alpha}[\mathbf{X}(x)] = [1 \quad x \quad x^2 \dots x^n] \begin{pmatrix} \frac{\Gamma(1)}{\Gamma(\alpha+1)}x^{\alpha} & 0 & \dots & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2+\alpha)}x^{\alpha} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{\Gamma(n+1)}{\Gamma(n+1+\alpha)}x^{\alpha} \end{pmatrix}. \tag{13}$$

Then, from the matrix form defined in Eq. (13), we have

$$J^{\alpha}[\mathbf{X}(x)] = \mathbf{X}(x)\boldsymbol{\Psi}(x), \tag{14}$$

where,

$$\boldsymbol{\Psi}(x) = \begin{pmatrix} \frac{\Gamma(1)}{\Gamma(\alpha+1)}x^{\alpha} & 0 & \dots & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2+\alpha)}x^{\alpha} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{\Gamma(n+1)}{\Gamma(n+1+\alpha)}x^{\alpha} \end{pmatrix}. \tag{15}$$

Now, if we define the integration of Eq. (11) as

$$J^{\alpha}[u(x)] = J^{\alpha}[\mathbf{X}(x)\mathbf{D}^{\mathbf{T}}\mathbf{Z}] = [J^{\alpha}\mathbf{X}(x)]\mathbf{D}^{\mathbf{T}}\mathbf{Z}, \tag{16}$$

then, from the relations in (14) and (16), we conclude the operational matrix of integration J^{α} as

$$J^{\alpha}[u(x)] = \mathbf{X}(x)\boldsymbol{\Psi}(x)\mathbf{D}^{\mathbf{T}}\mathbf{Z}. \tag{17}$$

For the part $\lambda_1 \int_0^x \frac{u^{(\alpha)}(t)}{\sqrt{x-t}}dt$ in the above equation, and using the formula given in [26], we have

$$\int_0^x \frac{t^n}{\sqrt{x-t}}dt = \frac{\sqrt{\pi}x^{(\frac{1}{2}+n)}\Gamma(n+1)}{\Gamma(n+\frac{3}{2})},$$

By using Eq. (11), we get

$$\lambda_1 \int_0^x \frac{u^{(\alpha)}(t)}{\sqrt{x-t}}dt = \lambda_1 \left[\int_0^x \frac{\mathbf{X}(t)}{\sqrt{x-t}}dt \right] \mathbf{D}^{\mathbf{T}}\mathbf{Z} = \lambda_1 O_x \mathbf{D}^{\mathbf{T}}\mathbf{Z}, \tag{18}$$

where

$$O_x = \left[\frac{\sqrt{\pi}x^{\frac{1}{2}}\Gamma(1)}{\Gamma(\frac{3}{2})} \quad \frac{\sqrt{\pi}x^{\frac{3}{2}}\Gamma(2)}{\Gamma(\frac{5}{2})} \quad \dots \quad \frac{\sqrt{\pi}x^{(\frac{1}{2}+n)}\Gamma(n+1)}{\Gamma(n+\frac{3}{2})} \right].$$

Now by applying the Riemann–Liouville fractional integral operator J^{α} into (18), we have the following relation

$$J^{\alpha}[\lambda_1 O_x \mathbf{D}^{\mathbf{T}}\mathbf{Z}] = \lambda_1 \mathbf{O}_x \mathbf{D}^{\mathbf{T}}\mathbf{Z}, \tag{19}$$

where $\mathbf{O}_x = J^{\alpha}[O_x]$. Note that, for the part $\lambda_2 \int_0^x K(x,t)u^{(\alpha)}(t)dt$, we have

$$\lambda_2 \int_0^x K(x,t)u^{(\alpha)}(t)dt = \lambda_2 \left[\int_0^x K(x,t)\mathbf{X}(t)dt \right] \mathbf{D}^{\mathbf{T}}\mathbf{Z} = \lambda_2 S_x \mathbf{D}^{\mathbf{T}}\mathbf{Z}, \tag{20}$$

where

$$S_x = \left[\int_0^x K(x,t)dt \quad \int_0^x K(x,t)tdt \quad \dots \quad \int_0^x K(x,t)t^n dt \right].$$

By applying the Riemann–Liouville fractional integral operator J^α into (20), we get the relation

$$J^\alpha \left[\lambda_2 S_x \mathbf{D}^\mathbf{T} \mathbf{Z} \right] = \lambda_1 \mathbf{S}_x \mathbf{D}^\mathbf{T} \mathbf{Z}, \tag{21}$$

where $\mathbf{S}_x = J^\alpha(S_x)$. Again, applying the operational matrix of integration J^α into (1), we get the relation

$$J^\alpha \left[u^\alpha(x) \right] = J^\alpha \left[u(x) \right] + J^\alpha \left[\lambda_1 \int_0^x \frac{u(t)}{\sqrt{x-t}} dt \right] + J^\alpha \left[\lambda_2 \int_0^x K(x,t)u(t)dt \right] + J^\alpha \left[h(x) \right]. \tag{22}$$

Now, substituting the matrix forms given in Eqs. (17), (19), and (21) into (22), we conclude the following equation

$$u(x) = u(0) + J^\alpha \left[u(x) \right] + J^\alpha [\lambda_1 O_x \mathbf{D}^\mathbf{T} \mathbf{Z}] + J^\alpha \left[\lambda_2 S_x \mathbf{D}^\mathbf{T} \mathbf{Z} \right] + \mathbf{R}(x). \tag{23}$$

Thus,

$$\mathbf{X}(x)\mathbf{D}^\mathbf{T}\mathbf{Z} - \mathbf{X}(x)\Psi(x)\mathbf{D}^\mathbf{T}\mathbf{Z} - \lambda_1 \mathbf{O}_x \mathbf{D}^\mathbf{T}\mathbf{Z} - \lambda_2 \mathbf{S}_x \mathbf{D}^\mathbf{T}\mathbf{Z} = u(0) + \mathbf{R}(x), \tag{24}$$

where $\mathbf{R}(x) = J^\alpha[h(x)]$, and so

$$\left[\mathbf{X}(x)\mathbf{D}^\mathbf{T} - \mathbf{X}(x)\Psi(x)\mathbf{D}^\mathbf{T} - \lambda_1 \mathbf{O}_x \mathbf{D}^\mathbf{T} - \lambda_2 \mathbf{S}_x \mathbf{D}^\mathbf{T} \right] \mathbf{Z} = \mathbf{E}(x), \tag{25}$$

where $\mathbf{E}(x) = u(0) + \mathbf{H}(x)$.

By using the collocation points $\{x_i : 0 \le i \le n\}$ in), the matrix $\mathbf{V} = \mathbf{V}_{(n+1)\times(n+1)}$ will be obtained, where

$$\mathbf{V} = \mathbf{X}(x)\mathbf{D}^\mathbf{T} - \mathbf{X}(x)\,\Psi(x)\mathbf{D}^\mathbf{T} - \lambda_1 \mathbf{O}_x \mathbf{D}^\mathbf{T} - \lambda_2 \mathbf{S}_x \mathbf{D}^\mathbf{T}. \tag{26}$$

The collocation points that we consider here are the roots of Chebyshev polynomials given by

$$x_i = \frac{1}{2} + \frac{1}{2\cos\left((2i+1)\frac{\pi}{2n}\right)}, \quad i = 0, 1 \dots n. \tag{27}$$

So, the main matrix Eq. corresponding to Eq. (1) can be formed as augmented matrix

$$\mathbf{VZ} = \mathbf{E} \quad \text{or} \quad [\mathbf{V}; \mathbf{E}]. \tag{28}$$

For the initial conditions given in Eq. (2) and based on Eq. (11), we can obtain the corresponding matrix forms as

$$\mathbf{X}(\delta)\mathbf{D}^\mathbf{T}\mathbf{Z} = [u_i], \quad 0 \le \delta \le R, \quad i = 0, 1, \dots, m. \tag{29}$$

Hence, we have

$$\Upsilon_i \mathbf{Z} = [u_i], \quad i = 0, 1, \dots, m, \tag{30}$$

where,

$$\Upsilon_i = \mathbf{X}(\delta)\mathbf{D}^\mathbf{T}. \quad i = 0, 1, \dots, m. \tag{31}$$

Combining $[\mathbf{V}, \mathbf{E}]$ and using the Gauss elimination method, we obtain $[\Upsilon_i, u_i]$. So, the new system will be of the following form $[\tilde{\mathbf{V}}, \tilde{\mathbf{E}}]$

$$[\tilde{\mathbf{V}}, \tilde{\mathbf{E}}] = \begin{pmatrix} \mathbf{V} & , & \mathbf{E} \\ \Upsilon_i & , & \mathbf{u}_i \end{pmatrix}, \tag{32}$$

which can be easily solved.

3.2. BSSD method

Let us constitute the second method by applying operational matrix of derivative with applications. We want to get an approximate solution, namely Bernstein series solution obtained by operational matrix of differentiation (BSSD), for the proposed problem

$$u(x) = \mathbf{B}_n(x)\mathbf{C}, \tag{33}$$

where $\mathbf{C}$ is the unknown coefficient matrix. A Similar argument to the BSSI method, the BSSD solution and its fractional derivative can be written as

$$u(x) = \mathbf{X}(x)\mathbf{D}^\mathbf{T}\mathbf{C}. \tag{34}$$

Then

$$D^{\alpha} u_N(x) = [D^{\alpha} \mathbf{X}(x)]\, \mathbf{D}^{\mathbf{T}} \mathbf{C}. \tag{35}$$

By using the Caputo definition in (7), the relation between the matrix $\mathbf{X}(x)$ and its derivative $D^{\alpha}[\mathbf{X}(x)]$ can be introduced as

$$D^{\alpha}[\mathbf{X}(x)] = \begin{bmatrix} 0 & \frac{\Gamma(2)}{\Gamma(2-\alpha)} x^{1-\alpha} & \frac{\Gamma(3)}{\Gamma(3-\alpha)} x^{2-\alpha} & \cdots & \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} x^{n-\alpha} \end{bmatrix}. \tag{36}$$

The Equation defined in (36) can be written as

$$D^{\alpha}[\mathbf{X}(x)] = [1 \quad x \quad x^2 \ldots x^n] \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2-\alpha)} x^{-\alpha} & 0 & \cdots & 0 \\ 0 & 0 & \frac{\Gamma(3)}{\Gamma(3-\alpha)} x^{-\alpha} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} x^{-\alpha} \end{pmatrix}. \tag{37}$$

So, we have

$$D^{\alpha}[\mathbf{X}(x)] = \mathbf{X}(x)\boldsymbol{\Omega}(x), \tag{38}$$

where,

$$\boldsymbol{\Omega}(x) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2-\alpha)} x^{-\alpha} & 0 & \cdots & 0 \\ 0 & 0 & \frac{\Gamma(3)}{\Gamma(3-\alpha)} x^{-\alpha} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} x^{-\alpha} \end{pmatrix}. \tag{39}$$

Hence, we can write the relation in (35) as

$$u^{(\alpha)}(x) = \mathbf{X}(x)\boldsymbol{\Omega}(x)\mathbf{D}^{\mathbf{T}}\mathbf{C}. \tag{40}$$

Now, using the formula (given in [26]),

$$\int_0^x \frac{t^n}{\sqrt{x-t}} dt = \frac{\sqrt{\pi} x^{(\frac{1}{2}+n)} \Gamma(n+1)}{\Gamma(n+\frac{3}{2})},$$

the part $\lambda_1 \int_0^x \frac{u_N^{(\alpha)}(t)}{\sqrt{x-t}} dt$ in Eq. (40) can be simplified as

$$\lambda_1 \int_0^x \frac{u_N^{(\alpha)}(t)}{\sqrt{x-t}} dt = \lambda_1 \left[\int_0^x \frac{\mathbf{X}(t)}{\sqrt{x-t}} dt \right] \boldsymbol{\Omega}(x)\mathbf{D}^{\mathbf{T}}\mathbf{C} = \lambda_1 Q_x \boldsymbol{\Omega}(x)\mathbf{D}^{\mathbf{T}}\mathbf{C}, \tag{41}$$

where,

$$Q_x = \left[\frac{\sqrt{\pi} x^{\frac{1}{2}} \Gamma(1)}{\Gamma(\frac{3}{2})} \quad \frac{\sqrt{\pi} x^{\frac{3}{2}} \Gamma(2)}{\Gamma(\frac{5}{2})} \quad \cdots \quad \frac{\sqrt{\pi} x^{(\frac{1}{2}+n)} \Gamma(n+1)}{\Gamma(n+\frac{3}{2})} \right].$$

Similarly,

$$\lambda_2 \int_0^x K(x,t) u^{(\alpha)}(t) dt = \lambda_2 \left[\int_0^x K(x,t)\mathbf{X}(t) dt \right] \boldsymbol{\Omega}(x)\mathbf{D}^{\mathbf{T}}\mathbf{C} = \lambda_2 S_x \boldsymbol{\Omega}(x)\mathbf{D}^{\mathbf{T}}\mathbf{C}, \tag{42}$$

where,

$$S_x = \left[\int_0^x K(x,t) dt \quad \int_0^x K(x,t) t dt \quad \cdots \quad \int_0^x K(x,t) t^n dt \right].$$

Substituting Eqs. (40), (41) and (42) into Eq. (1), we obtain the substantial matrix equation

$$\left[\mathbf{X}(x)\boldsymbol{\Omega}(x)\mathbf{D}^{\mathbf{T}} - \mathbf{X}(x)\mathbf{D}^{\mathbf{T}} - \lambda_1 Q_x \boldsymbol{\Omega}(x)\mathbf{D}^{\mathbf{T}} - \lambda_2 S_x \boldsymbol{\Omega}(x)\mathbf{D}^{\mathbf{T}} \right] \mathbf{C} = \mathbf{H}(x). \tag{43}$$

By substituting the collocation points $\{x_i : 0 \le i \le n\}$ into Eq. (43), the matrix $\mathbf{V}_{(n+1)\times(n+1)}$ will be obtained. The collocation points are given by the following relation, where

$$x_i = \frac{i}{n}, \quad 0 \le i \le n. \tag{44}$$

Hence, Equation in (43) become,

$$\left[\mathbf{X}\Omega\mathbf{D}^{\mathbf{T}} - \mathbf{X}\mathbf{D}^{\mathbf{T}} - \lambda_1 Q\Omega\mathbf{D}^{\mathbf{T}} - \lambda_2 S\Omega\mathbf{D}^{\mathbf{T}}\right]\mathbf{C} = \mathbf{H}, \tag{45}$$

which can be written as a fundamental matrix form given by,

$$\mathbf{W}\mathbf{C} = \mathbf{H} \quad \text{or} \quad [\mathbf{W}; \mathbf{H}], \tag{46}$$

where,

$$\mathbf{W} = \mathbf{X}\Omega\mathbf{D}^{\mathbf{T}} - \mathbf{X}\mathbf{D}^{\mathbf{T}} - \lambda_1 Q\Omega\mathbf{D}^{\mathbf{T}} - \lambda_2 S\Omega\mathbf{D}^{\mathbf{T}}.$$

The initial conditions in (2) can be written in the matrix forms as follows

$$\mathbf{X}(\delta)\mathbf{D}^{\mathbf{T}}\mathbf{C} = [u_i], \quad 0 \le \delta \le R, \quad i = 0, 1,$$

or

$$\Upsilon_i\mathbf{C} = [u_i] = [u_{i0} \quad u_{i1} \quad \dots \quad u_{in}], \quad i = 0, 1, \dots, m. \tag{47}$$

Combining $[\mathbf{W}, \mathbf{H}]$ and $[\Upsilon_i, u_i]$, we obtain a new system $[\bar{\mathbf{W}}, \bar{\mathbf{H}}]$,

$$[\tilde{\mathbf{V}}, \tilde{\mathbf{W}}] = \begin{pmatrix} \mathbf{V} & , & \mathbf{H} \\ \Upsilon_i & , & \mathbf{u}_i \end{pmatrix}. \tag{48}$$

If the matrix $\breve{\mathbf{W}}$ is square matrix and invertible, then we can find the unknowns $\mathbf{C} = [c_0, \quad c_1 \quad \cdots \quad c_n]$ by

$$\mathbf{C} = (\breve{\mathbf{W}})^{-1}\breve{\mathbf{H}}.$$

Note that, we omit the stability analysis and residual correction procedure for the method as it can be done in the same fashion to the stability analysis of the BSSI given in Section 3.3.

3.3. Stability analysis of BSSI and residual analysis

In this section, we will constitute the stability estimation as follows. Suppose that the solution of the perturbing system is u_n^p, i.e., u_n^p is the solution of the following perturbing system:

$$\begin{aligned} u^{\alpha}(x) &= u(x) + \lambda_1 \int_0^x \frac{u^{(\alpha)}(t)}{\sqrt{x-t}}dt + \lambda_2 \int_0^x K(x,t)u^{(\alpha)}(t)dt + h(x), \\ u^{(i)}(\delta) &= u_i + \varepsilon. \end{aligned}$$

Then, performing the proposed method yields the following:

$$\bar{\mathbf{V}}C = \bar{\mathbf{Z}} + \Delta\bar{\mathbf{Z}}. \tag{49}$$

Let $\mathbf{C}^p =$ be the perturbed solution of (49). Then, $\mathbf{C}$ is bounded as follows [27]

$$\frac{\|\Delta\mathbf{C}\|}{\|\mathbf{C}\|} \le cond(\bar{\mathbf{V}})\frac{\|\Delta\bar{\mathbf{Z}}\|}{\|\bar{\mathbf{Z}}\|}.$$

Similarly, for the error that may occur in $\bar{\mathbf{V}}$ as a result of arithmetic operations, we consider the perturbed problem

$$\left(\bar{\mathbf{V}} + \Delta\bar{\mathbf{V}}\right)\mathbf{C} = \bar{\mathbf{Z}} + \Delta\bar{\mathbf{Z}}. \tag{50}$$

As the same notation, the change in $\mathbf{C}$ caused by perturbing the initials and arithmetic operations is bounded above as follows [27].

$$\frac{\|\Delta\mathbf{C}\|}{\|\mathbf{C}\|} \le \frac{cond(\bar{\mathbf{V}})}{1 - cond(\bar{\mathbf{V}})\frac{\|\Delta\bar{\mathbf{V}}\|}{\|\bar{\mathbf{V}}\|}}\left(\frac{\|\Delta\bar{\mathbf{V}}\|}{\|\bar{\mathbf{V}}\|} + \frac{\|\Delta\bar{\mathbf{Z}}\|}{\|\bar{\mathbf{Z}}\|}\right).$$

Thus, for the given Eq. (49) and based on the BSSI we have,

$$\begin{aligned} \left|u_n(x) - u_n^p(x)\right| &= |\mathbf{B}_n(x)\left(\mathbf{C} - (\mathbf{C} + \Delta\mathbf{C})\right)| \\ &\le \|\mathbf{B}_n(x)\| \, \|\Delta\mathbf{C}\| \\ &\le \|\mathbf{B}_n(x)\| \, cond(\bar{\mathbf{V}})\frac{\|\Delta\bar{\mathbf{Z}}\| \, \|\mathbf{C}\|}{\|\bar{\mathbf{Z}}\|}. \end{aligned} \tag{51}$$

A Similar conclusions can be achieved for Eq. (50)

To construct the residual correction procedure for the problem, let R_n be defined as follows,

$$R_n(x) := u_n^{\alpha}(x) - u_n(x) - \lambda_1 \int_0^x \frac{u_n^{(\alpha)}(t)}{\sqrt{x-t}}dt - \lambda_2 \int_0^x K(x,t)u_n^{(\alpha)}(t)dt.$$

Then, adding and subtracting the term R_n from Eq. (1) gives the following problem for the absolute error

$$e_n^{\alpha}(x) = e_n(x) + \lambda_1 \int_0^x \frac{e_n^{(\alpha)}(t)}{\sqrt{x-t}}dt + \lambda_2 \int_0^x K(x,t)e_n^{(\alpha)}(t)dt + h(x), \tag{52}$$

where $e_n = u - u_n$ with the following initial condition

$$e_n^{(\alpha)}(\delta) = 0. \tag{53}$$

By applying the method into Eq. (52) with the condition (53), we can obtain an approximate solution, which is denoted by $e_{n,m}$, for the absolute error where m is the degree of approximation.

Note that $u_{n,m} := u_n + e_{n,m}$ is another approximate solution, called corrected BSSI solution, and its error function is $e_{n,m}$. If $\left\|e_n - e_{n,m}\right\| < \|y - e_n\|$, then $u_{n,m}$ is a better approximation than u_n in the norm. On the other hand, we can estimate e_n by $e_{n,m}$ whenever $\left\|e_n - e_{n,m}\right\| < \varepsilon$ is small.

4. Numerical implementation

In this section, four examples are considered to illustrate the properties and effectiveness of the methods. Three of them have the smooth exact solutions and the last one has the non-smooth exact solution. We compare the results with some known methods.

Example 1

Consider the fractional integro-differential equation,

$$u^{(0.5)}(x) = \frac{8x^{1.5} - 6x^{0.5}}{3\sqrt{\pi}} + \frac{x}{12} + \int_0^1 xtu(t)dt, \tag{54}$$

with an initial condition,

$$u(0) = 0.$$

The exact solution (see [28,29]) of this problem is given by

$$u(x) = x^2 - x.$$

By applying the BSSI method into Eq. (54), we get the relation

$$J^{0.5}\left[u^{(0.5)}(x)\right] = J^{0.5}\left[\int_0^1 xtu(t)dt\right] + J^{0.5}\left[\frac{8x^{1.5} - 6x^{0.5}}{3\sqrt{\pi}} + \frac{x}{12}\right], \tag{55}$$

which can be written as

$$u(x) - J^{0.5}\left[\int_0^1 xtu(t)dt\right] = u(0) + J^{0.5}\left[\frac{8x^{1.5} - 6x^{0.5}}{3\sqrt{\pi}} + \frac{x}{12}\right]. \tag{56}$$

The fundamental matrix equation for Eq. (54) is obtained as

$$\left[\mathbf{X}(x)\mathbf{D^T} - \mathbf{X}(x)\boldsymbol{\Psi}(x)\mathbf{D^T} - \mathbf{S}_x\mathbf{D^T}\right]\mathbf{Z} = \mathbf{E}(x). \tag{57}$$

By substituting the collocation nodes (27) into (57) when $n = 2$, the following matrices are obtained,

$$\boldsymbol{\Psi}(x) = \begin{pmatrix} \frac{\Gamma(1)}{\Gamma(1.5)}x^{0.5} & 0 & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2.5)}x^{0.5} & 0 \\ 0 & 0 & \frac{\Gamma(3)}{\Gamma(3.5)}x^{0.5} \end{pmatrix}, \quad \mathbf{D^T} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{pmatrix},$$

$$\mathbf{Z} = \begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.5 \\ 0 \end{bmatrix}, \quad \mathbf{V} = \begin{pmatrix} 0.7250 & 0.2429 & 0.0109 \\ 0.7250 & 0.2429 & 0.0109 \\ -0.0279 & 0.1511 & 0.5802 \end{pmatrix}, \quad \mathbf{E} = \begin{bmatrix} 0.1214 \\ 0.1214 \\ 0.0755 \end{bmatrix}.$$

Then, by using Eq. (11), the BSSI solution is obtained by the following

$$u(x) = \begin{bmatrix} 1 & x & x^2 \end{bmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{pmatrix} \begin{bmatrix} 0 \\ -0.5 \\ 0 \end{bmatrix} = x^2 - x,$$

which gives the exact solution.

Table 1
A comparison between the BSS algorithms and other methods for Example 1.

x	SLM [29]	PLM [29]	OHAM [28]	BSS methods
0.0	3.48×10^{-5}	1.08×10^{-5}	0	0
0.1	1.67×10^{-5}	9.10×10^{-5}	0	0
0.2	2.74×10^{-6}	7.88×10^{-5}	0	0
0.3	7.26×10^{-6}	7.13×10^{-5}	0	0
0.4	1.32×10^{-5}	6.84×10^{-5}	0	0
0.5	1.51×10^{-5}	7.01×10^{-5}	5.55×10^{-17}	0
0.6	1.31×10^{-5}	7.62×10^{-5}	0	0
0.0	6.99×10^{-6}	8.67×10^{-5}	0	0
0.8	3.15×10^{-6}	1.01×10^{-4}	1.11×10^{-16}	0
0.9	1.73×10^{-5}	1.20×10^{-4}	1.11×10^{-16}	0
1.0	3.55×10^{-6}	1.43×10^{-4}	1.11×10^{-16}	0

If we apply the BSSD method to solve the problem in Eq. (54), the fundamental matrix equation for Eq. (54) is obtained as

$$\left[\mathbf{X}\Omega\mathbf{D}^{\mathbf{T}} - S\mathbf{D}^{\mathbf{T}}\right]\mathbf{C} = \mathbf{H}. \tag{58}$$

For $n = 2$, we get the following matrices,

$$\mathbf{\Omega}(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2.5)}x^{-0.5} & 0 \\ 0 & 0 & \frac{\Gamma(3)}{\Gamma(3.5)}x^{-0.5} \end{pmatrix}.$$

$$\mathbf{W} = \begin{pmatrix} -0.7915 & -0.7915 & -0.9697 \\ 0.6705 & 0.6705 & -0.4301 \\ 0.0477 & 0.0477 & 0.9730 \end{pmatrix}, \quad \mathbf{D}^{\mathbf{T}} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{pmatrix}, \quad \mathbf{H} = \begin{bmatrix} -0.3352 \\ -0.3352 \\ -0.0728 \end{bmatrix}.$$

Using collocation points in (44) with $n = 2$ and substituting these points to Eq. (58), we have

$$\mathbf{C} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.5 \\ 0 \end{bmatrix}.$$

Now, from Eq. (11) the BSSD solution is obtained as

$$u(x) = [1 \quad x \quad x^2]\begin{pmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{pmatrix}\begin{bmatrix} 0 \\ -0.5 \\ 0 \end{bmatrix} = x^2 - x,$$

which gives again the exact solution. Note that, for both methods the exact solution is obtained for $n = 2$. We compare these results with optimal homotopy asymptotic method (OHAM), Standard Least Squares Method (SLM) and Perturbed Least Squares Method (PLM). All of them ended up with the exact solution whereas the methods SLM [29], PLM [29] and OHAM [28] ended by an approximate solution within a good accuracy, however not exact for $n = 2$ and for this specific example. Table 1 presents the error bound achieved using the BSSI and BSSD methods and compared with other techniques.

Example 2

Let us consider the fractional integro-differential equation [30]

$$D^{(0.25)}u(x) = \frac{1}{2}\int_0^x \frac{u(t)}{\sqrt{x-t}}dt + \frac{1}{3}\int_0^1 (x-t)u(t)dt + g(x), \tag{59}$$

with an initial condition is $u(0) = 0$, where

$$g(x) = \frac{\Gamma(3)}{\Gamma(2.75)}x^{1.75} + \frac{\Gamma(4)}{\Gamma(3.75)}x^{2.75} - \frac{\sqrt{\pi}\Gamma(3)}{2\Gamma(7/2)}x^{5/2} - \frac{\sqrt{\pi}\Gamma(4)}{2\Gamma(9/2)}x^{7/2} - \frac{7x}{36} + \frac{3}{20}.$$

The exact solution of this formulation is

$$u(x) = x^2 + x^3.$$

Consider $n = 3$, and apply the BSSI method, we get the following

$$J^{0.25}\left[u^{(0.25)}(x)\right] = J^{0.25}\left[\frac{1}{2}\int_0^x \frac{u(t)}{\sqrt{x-t}}dt\right] + J^{0.25}\left[\frac{1}{3}\int_0^1 (x-t)u(t)dt\right] + J^{0.25}\left[g(x)\right],$$

Table 2
A comparison between our methods fro $n = 3$, and other methods for Example 2.

x	CAS method [30]	Legendre wavelets method [30]	BSSD method	Exact
0.0	0.0400000000	0.0000000000	0.0000000000	0.0000000000
1/6	0.0900000000	0.0600000000	0.0324074074	0.0324074074
2/6	0.1900000000	0.1800000000	0.1481481481	0.1481481481
3/6	0.4000000000	0.4000000000	0.3750000000	0.3750000000
4/6	0.7200000000	0.7800000000	0.7407407407	0.7407407407
5/6	1.1800000000	1.2800000000	1.2731481480	1.2731481480

and the matrix equation

$$\left[\mathbf{X}(x)\mathbf{D^T} - \frac{1}{2}\mathbf{O}_x\mathbf{D^T} - \frac{1}{3}\mathbf{S}_x\mathbf{D^T}\right]\mathbf{Z} = \mathbf{E}(x),$$

where,

$$\mathbf{\Psi}(x) = \begin{pmatrix} \frac{\Gamma(1)}{\Gamma(1.5)}x^{0.25} & 0 & 0 & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2.5)}x^{0.25} & 0 & 0 \\ 0 & 0 & \frac{\Gamma(3)}{\Gamma(3.5)}x^{0.25} & 0 \\ 0 & 0 & 0 & \frac{\Gamma(4)}{\Gamma(4.5)}x^{0.25} \end{pmatrix},$$

$$\mathbf{D^T} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} -0.2472 & -0.2201 & 0.0956 & 0.8191 \\ -0.2444 & -0.0513 & 0.3249 & 0.1155 \\ -0.1587 & 0.2518 & 0.1860 & 0.0271 \\ 0.6157 & 0.2708 & 0.0634 & 0.0313 \end{pmatrix},$$

$$\mathbf{Z} = \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{3} \\ 2 \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} 1.1786 \\ 0.6255 \\ 0.1888 \\ 0.0716 \end{bmatrix}.$$

Thus, the BSSI solution reveals the solution as $x^3 + x^2$, which is the exact.

If we apply the BSSD method to the problem, the fundamental matrix equation for Eq. (54) is obtained as

$$\left[\mathbf{X}(x)\mathbf{\Omega}(x)\mathbf{D^T} - \frac{1}{2}Q_x\mathbf{\Omega}(x)\mathbf{D^T} - \frac{1}{3}S_x\mathbf{\Omega}(x)\mathbf{D^T}\right]\mathbf{C} = \mathbf{H}(x). \tag{60}$$

By applying the method with $n = 3$, we get the following matrices

$$\mathbf{\Omega}(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2.5)}x^{-0.5} & 0 & 0 \\ 0 & 0 & \frac{\Gamma(3)}{\Gamma(3.5)}x^{-0.5} & 0 \\ 0 & 0 & 0 & \frac{\Gamma(4)}{\Gamma(4.5)}x^{-0.5} \end{pmatrix}.$$

$$\mathbf{W} = \begin{pmatrix} -1.11457 & -1.244036 & -1.2272 & -0.43640 \\ -.3697304 & -.240228 & 0.370244 & 0.2735 \\ -.457121 & .2139 & .27826 & 0.058117 \\ 0.8066644211 & 0.08699 & 0.9730 & 0.063659 \end{pmatrix}, \quad \mathbf{D^T} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix}.$$

Putting in the discretized points to Eq. (60) yields the following matrices,

$$\mathbf{H} = \begin{bmatrix} 1.4609 \\ 0.8213 \\ 0.2667 \\ 0.1466 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 6.4850E-20 \\ 0.3334 \\ 1.9999 \end{bmatrix}.$$

Hence, the BSSD solution is obtained as $1.9455(E - 19)x + 1.0x^3 + 0.9999x^2$.

In Table 2, we provide some numerical experiments for the BSSD solution when $n = 3$ compared with some known methods. It is clear that the BSS methods gives us more accurate results than the CAS wavelets and Legendre wavelets methods [30] for this problem. The approximate solutions obtained by BSS methods for $n = 2$ and $n = 3$ are depicted in Fig. 1. Table 3 shows the stability results based on the BSSD method.

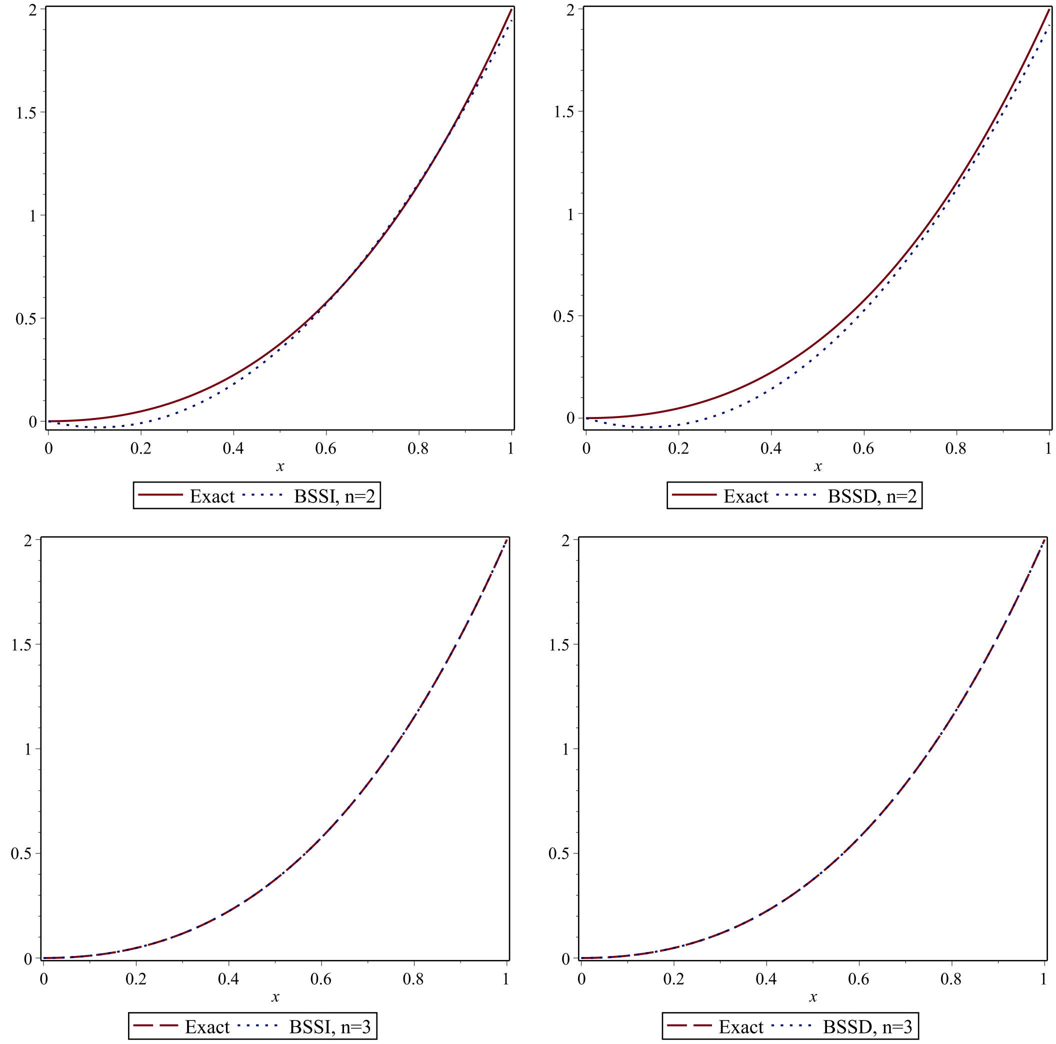

Fig. 1. The BSS solutions for $n=2$ and $n=3$ with the exact solution for Example 2.

Table 3
Stability results of the system obtained by BSSD method for Example 2.

	$n=2$	$n=3$	$n=4$
$cond(\bar{W})$	12.82	15.24	28.07
$\|\Delta C\|$	8.00×10^{-10}	1.19×10^{-16}	1.07×10^{-9}
$\|C\|$	1.92	2.00	2.00
$\|\Delta H\|$	10^{-16}	10^{-16}	10^{-16}
$\|H\|$	1.38	1.46	1.49
Upper bound obtained by (51)	7.12×10^{-15}	1.66×10^{-14}	5.99×10^{-14}
$\left\|u_n - u_n^p\right\|$	1.0×10^{-16}	2.0×10^{-16}	3.0×10^{-16}

Example 3

Now, we consider the following fractional integro-differential equation [31]

$$D^{\frac{1}{3}}u(x) = p(x)u(x) + \int_0^x \frac{u(t)}{\sqrt{x-t}}dt + g(x), \tag{61}$$

where the initial condition is given to be $u(0)=0$, and

$$g(x) = \frac{6}{\Gamma(\frac{11}{3})}x^{\frac{8}{3}} + (\frac{32}{35} - \frac{\Gamma(\frac{1}{2})\Gamma(\frac{7}{3})}{\Gamma(\frac{17}{6})})x^{11/6} + \Gamma(\frac{7}{3})x, \quad p(x) = \frac{-32}{35}x^{\frac{1}{2}}.$$

Table 4
The absolute error of the BSSD solution for $n = 3, 6, 8$ for Example 3.

x	$n = 3$	$n = 6$	$n = 8$
0.0	0	0	0
0.1	5.54×10^{-4}	1.58×10^{-3}	1.51×10^{-4}
0.2	6.93×10^{-3}	1.35×10^{-4}	4.69×10^{-4}
0.3	1.07×10^{-2}	1.10×10^{-3}	4.87×10^{-4}
0.4	1.10×10^{-3}	1.02×10^{-4}	3.75×10^{-4}
0.5	8.61×10^{-3}	1.16×10^{-3}	3.68×10^{-4}
0.6	4.55×10^{-3}	1.10×10^{-3}	1.73×10^{-4}
0.7	5.19×10^{-4}	8.54×10^{-5}	4.83×10^{-4}
0.8	1.69×10^{-3}	6.77×10^{-4}	1.16×10^{-4}
0.9	1.68×10^{-4}	5.10×10^{-4}	2.23×10^{-4}
1.0	7.12×10^{-3}	3.53×10^{-6}	2.73×10^{-6}

Table 5
The correction procedure of the BSSD solution for $n = 3$, and $m = 10, 12, 14$ for Example 3.

x	$\boldsymbol{e}_{3,3}$	$\boldsymbol{e}_{3,3}^{10,10}$	$\boldsymbol{e}_{3,3}^{12,12}$	$\boldsymbol{e}_{3,3}^{14,14}$
0.0	0	0	0	0
0.1	5.54×10^{-4}	3.01×10^{-4}	1.23×10^{-4}	8.59×10^{-5}
0.2	6.93×10^{-3}	2.70×10^{-4}	9.20×10^{-5}	1.19×10^{-5}
0.3	1.07×10^{-2}	1.34×10^{-5}	1.16×10^{-4}	1.22×10^{-5}
0.4	1.10×10^{-3}	2.18×10^{-4}	1.78×10^{-4}	1.00×10^{-5}
0.5	8.61×10^{-3}	2.54×10^{-4}	1.24×10^{-4}	9.42×10^{-5}
0.6	4.55×10^{-3}	5.15×10^{-5}	5.90×10^{-5}	5.88×10^{-5}
0.7	5.19×10^{-4}	1.86×10^{-4}	5.47×10^{-5}	8.16×10^{-5}
0.8	1.69×10^{-3}	2.32×10^{-4}	6.02×10^{-5}	3.87×10^{-5}
0.9	1.68×10^{-4}	1.21×10^{-4}	5.92×10^{-5}	1.74×10^{-5}
1.0	7.12×10^{-3}	2.14×10^{-5}	6.00×10^{-6}	5.00×10^{-6}

Table 6
Stability results of the system obtained by the BSSD method for Example 3.

	$n = 3$	$n = 6$	$n = 10$
$cond(\bar{W})$	16.7202	107.7649	1685.8626
$\|\Delta C\|$	2.2981×10^{-16}	3.0×10^{-20}	2.2×10^{-19}
$\|C\|$	1.9928	1.9999	2.0000
$\|\Delta H\|$	10^{-16}	10^{-16}	10^{-16}
$\|H\|$	2.0813	2.2989	2.3483
Upper bound obtained by (51)	1.6009×10^{-15}	9.3842×10^{-15}	1.4359×10^{-13}
$\left\|u_n - u_n^p\right\|$	2.5×10^{-16}	2.50×10^{-16}	3.5×10^{-16}

The exact solution here is defined as

$$u(x) = x^3 + x^{\frac{4}{3}}.$$

By applying the BSSD method for $n = 3$, $n = 6$, and $n = 8$, the results are tabulated in Table 4. We apply the residual correction procedure given in Section 3.3 to the problem for $n = 3$ and $m = 10, 12, 14$ where some of the numerical observations are presented in Fig. 2 and Table 5. We can say from Fig. 2 and Table 5 that the absolute error for $n = 3$ is estimated well by using the residual correction procedure. We get the corrected solutions for $n = 3$ and $m = 10, 12, 14$ and give the absolute errors in Fig. 3. It can be inferred from Fig. 3 that the corrected solution is better than those obtained from the BSSD solution for $n = 3$ and $m = 10, 12, 14$. We also show some stability results for the BSSD method in Table 6.

Example 4

Consider the fractional integral–differential equation

$$u^{(0.5)}(x) = u(x) + \int_0^x u(t)dt + \frac{\Gamma(3)}{\Gamma(2.5)}x^{1.5} - \frac{1}{3}x^3 - x^2, \tag{62}$$

with initial condition defined by

$$u(0) = 0.$$

The exact solution of this problem is [32]

$$u(x) = x^2.$$

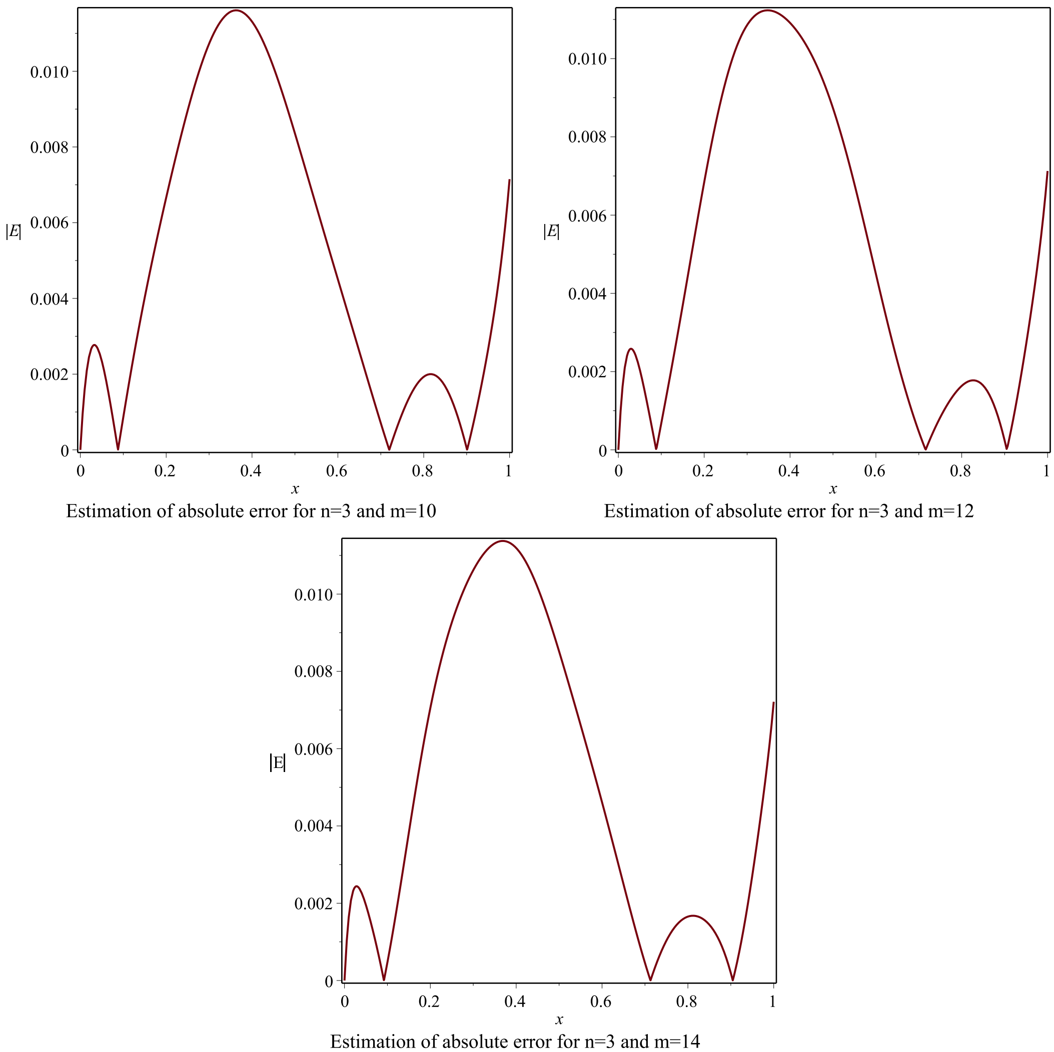

Fig. 2. Estimation of absolute error of the BSSD solutions for $n = 3$, and $m = 10, 12, 14$ for Example 3.

The fundamental matrix equation for Eq. (62) is obtained as

$$J^{0.5}\left[u^{(0.5)}(x)\right] = J^{0.5}\left[u(x)\right] + J^{0.5}\left[\int_0^x u(t)dt\right] + J^{0.5}\left[\frac{\Gamma(3)}{\Gamma(2.5)}x^{1.5} - \frac{1}{3}x^3 - x^2\right]. \tag{63}$$

This can be simplified as follows.

$$u(x) - J^{0.5}\left[u(x)\right] - J^{0.5}\left[\int_0^x u(t)dt\right] = u(0) + J^{0.5}\left[\frac{\Gamma(3)}{\Gamma(2.5)}x^{1.5} - \frac{1}{3}x^3 - x^2\right]. \tag{64}$$

Obtaining the matrix form yields

$$\left[\mathbf{X}(x)\mathbf{D}^{\mathbf{T}} - \mathbf{X}(x)\Psi(x)\mathbf{D}^{\mathbf{T}} - \mathbf{S}_x\mathbf{D}^{\mathbf{T}}\right]\mathbf{Z} = \mathbf{E}(x), \tag{65}$$

Using the initial condition, we have

$$\mathbf{X}(0)\mathbf{D}^{\mathbf{T}}\mathbf{C} = 0.$$

The augmented matrix is formed as

$$\mathbf{VC} = \mathbf{Z} \quad \text{or} \quad [\mathbf{V}; \mathbf{Z}]. \tag{66}$$

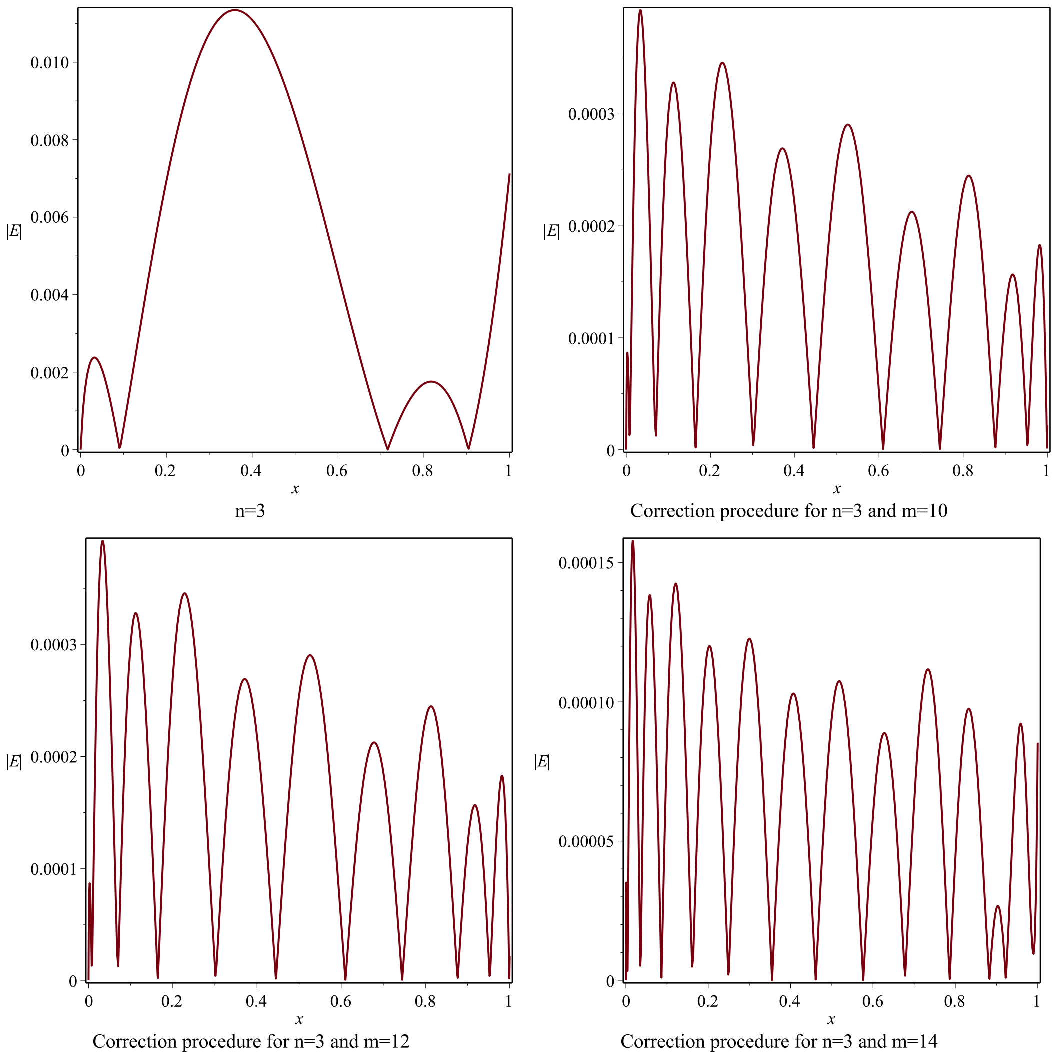

Fig. 3. Comparison the absolute error for $n = 3$ with the absolute errors obtained by the corrected BSSD solutions for $n = 3$, $m = 10, 12, 14$ for Example 3.

For $n = 2$ and $\alpha = 0.5$, we get the following matrices as

$$\Psi(x) = \begin{pmatrix} \frac{\Gamma(1)}{\Gamma(1.5)}x^{0.5} & 0 & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2.5)}x^{0.5} & 0 \\ 0 & 0 & \frac{\Gamma(3)}{\Gamma(3.5)}x^{0.5} \end{pmatrix}, \quad \mathbf{D^T} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{pmatrix},$$

$$\mathbf{Z} = \begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.5 \\ 0 \end{bmatrix}, \quad \mathbf{V} = \begin{pmatrix} 0.33869 & 0.33869 & -0.52661 \\ 0.171035 & 0.171035 & -0.33378 \\ 0.01630 & 0.01630 & 0.22469 \end{pmatrix}, \quad \mathbf{E} = \begin{bmatrix} 0.01630 \\ 0.01630 \\ 0.22469 \end{bmatrix}.$$

Then, by using Eq. (11), the BSSI solution is obtained as

$$u(x) = [1 \quad x \quad x^2] \begin{pmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{pmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = x^2,$$

which is the exact solution.

If we applying the BSSD method, then the fundamental matrix equation for Eq. (62) is obtained as

$$\left[\mathbf{X(x)}\boldsymbol{\Omega(x)}\mathbf{D^T} - \mathbf{X(x)D^T} - S_x\mathbf{D^T}\right]\mathbf{C} = \mathbf{H(x)}, \tag{67}$$

Table 7
The absolute error of the BSS methods for $n = 2$, and other methods for Example 4.

x	FEFs method $n = 4$ [32]	Legendre wavelets method $n = 6$ [33]	BSS methods $n = 2$
0.0	1.63×10^{-16}	5.55×10^{-16}	0
0.1	1.06×10^{-15}	6.66×10^{-16}	0
0.2	1.04×10^{-15}	9.15×10^{-16}	0
0.3	5.22×10^{-16}	1.27×10^{-15}	0
0.4	1.42×10^{-16}	1.63×10^{-15}	0
0.5	6.75×10^{-16}	2.04×10^{-15}	0
0.7	8.59×10^{-16}	2.52×10^{-15}	0
0.8	5.14×10^{-16}	3.27×10^{-15}	0
0.9	5.06×10^{-16}	3.77×10^{-15}	0

Now, for $n = 2$, we get the below matrices,

$$\mathbf{X}(x) = [1 \quad x \quad x^2], \quad \boldsymbol{\Omega}(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2.5)}x^{0.5} & 0 \\ 0 & 0 & \frac{\Gamma(3)}{\Gamma(3.5)}x^{0.5} \end{pmatrix},$$

$$\mathbf{C} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}, \quad \mathbf{D}^{\mathbf{T}} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{pmatrix},$$

Using the collocation points with $n = 2$,

$$x_i = [x_0 \quad x_1 \quad x_2] = [0 \quad 0.5 \quad 1],$$

we have,

$$\mathbf{C} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

From Eq. (11), we get an approximate solution

$$u(x) = [1 \quad x \quad x^2] \begin{pmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{pmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = x^2,$$

which is again the exact one.

Table 7 shows a comparison between the BSS methods with some known techniques such as the fractional-order Euler functions method and the Legendre wavelet method.

5. Conclusions

In this paper, we proposed two numerical techniques of operational matrices based on B-polynomials to solve a class of integro-differential equations of fractional order. The first method (the BSSI), is constituted of operational matrices of integral with applications, whereas the second one (the BSSD) depends on the operational matrices of derivative. We presented the residual correction procedure for the methods in order to estimate the absolute error and study the stability results based on the techniques. We also tested the proposed methods on some examples to demonstrate its efficiency. We compare the methods with some known results, where it is clearly the used algorithm showed more accurate results than those obtained by for example the optimal homotopy asymptotic method, standard and perturbed least squares method, CAS method and Legendre wavelets method and fractional-order Euler functions method for the considered examples. The numerical implementations indicate that these is a good agreement between the theoretical and numerical results. This been achieved in Tables 1–7 and Figs. 1–3. Throughout the presented examples, we provided the fundamental matrices of integration and differentiation $\boldsymbol{\Psi}(x)$, $\boldsymbol{\Omega}$, $\mathbf{D}$, $\mathbf{V}$, $\mathbf{Z}$, and $\mathbf{E}$ generated based on our techniques.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] Maleknejad K, Hashemizadeh E, Basirat B. Computational method based on Bernstein operational matrices for nonlinear Volterra–Fredholm–Ham- merstein integral equations. Commun Nonlinear Sci Numer Simul 2012;17(1):52–61.
[2] Baleanu D, Jajarmi A, Asad J, Blaszczyk T. The motion of a bead sliding on a wire in fractional sense. Acta Phys Polonica A 2017;131(6):1561–4.
[3] Baleanu D, Jajarmi A, Hajipour M. On the nonlinear dynamical systems within the generalized fractional derivatives with Mittag—Leffler kernel. Nonlinear Dynam 2018;94:397–414.
[4] Bracewell R, Riddle A. Inversion of fan-beam scans in radio astronomy. Agron J 1967;150:427.
[5] Macelwane J. Evidence on the interior of the earth derived from seismic sources. Intern Const Earth 1951;194:227–304.
[6] Kanwal R, Liu K. A Taylor expansion approach for solving integral equations. Int J Math Educ Sci Technol 1989;20:411–4.
[7] Ahmed Shams, Elzaki Tarig. On the comparative study integro–Differential equations using difference numerical methods. J King Saud Univ–Sci 2020;32:84–9.
[8] Mohammed D. Numerical solution of fractional integro-differential equations by least squares method and shifted Chebyshev polynomial. Math Probl Eng 2014;2014.
[9] Atanackovi T, Stankovic B. On a system of differential equations with fractional derivatives arising in rod theory. J Phys A: Math Gen 2004;37(4):1241.
[10] Mirzaee F, Samadyar N. Application of hat basis functions for solving two-dimensional stochastic fractional integral equations. Comput Appl Math 2018;37(4):4899–916.
[11] Mohammadi F. Wavelet Galerkin method for solving stochastic fractional differential equations. Fract Calc Appl Anal 2016;7(1):73–86.
[12] Mohammad M, Troune A. Implicit Riesz wavelets based-method for solving singular fractional integro-differential equations with applications to hematopoietic stem cell modeling. Chaos Solitons Fractals 2020;138:109991.
[13] Mohammad M, Troune A. Fractional nonlinear Volterra–Fredholm integral equations involving Atangana–Baleanu fractional derivative: framelet applications. Adv Difference Equ 2020;618.
[14] Chouhan D, Mishra V, Srivastava HM. Bernoulli wavelet method for numerical solution of anomalous infiltration and diffusion modeling by nonlinear fractional differential equations of variable order. Results Appl Math 2021.
[15] Alshbool MHT, Shatanawi W, Hashim I, Sarr M. Residual correction procedure with Bernstein polynomials for solving important systems of ordinary differential equations. Comput Mater Contin 2020;64:63–80.
[16] Khataybeh S, Hashim I, Alshbool MHT. Solving directly third-order ODEs using operational matrices of Bernstein polynomials method with applications to fluid flow. J King Saud Univ - Sci.
[17] Abel U, Siebert H. An improvement of the constant in Videnski?'s inequality for Bernstein polynomials. Georgian Math J 2020;27:1–7.
[18] Alshbool MH, Isik O, Hashim I. Fractional Bernstein series solution of fractional diffusion equations with error estimate. Axioms 2021;10.
[19] Isik O, Sezer M, Guney Z. A rational approximation based on Bernstein polynomials for high order initial and boundary values problems. Appl Math Comput 2011;217:9438–50.
[20] Isik O, Sezer M, Guney Z. Bernstein series solution of linear second-order partial differential equations with mixed conditions. Math Methods Appl Sci 2014;37:609–19.
[21] Baseri A, Babolian E, Abbasbandy S. Normalized Bernstein polynomials in solving space-time fractional diffusion equation. Appl Math Comput 2017;346(2017). http://dx.doi.org/10.1186/s13662-017-1401-1.
[22] ALshbool Mohammed, Hashim I. Bernstein polynomials for solving nonlinear stiff system of ordinary differential equations. AIP Conf Proc 2015;1678:060015.
[23] Sabatier J, Agrawal P, Machado JA. Advances in fractional calculus: theoretical developments and applications in physics and engineering. In: Theoretical developments and applications in physics and engineering. Netherlands: Springer; 2007.
[24] Miller KS, Ross B. An introduction to the fractional calculus and fractional differential equations. New York, USA: Wiley-Interscience, John Wiley and Sons; 1993.
[25] Diethelm K, Ford N, Freed A. Algorithms for the fractional calculus: a selection of numerical methods. Comput Methods Appl Mech Engrg 2005;194:743—-773.
[26] Singh V, Pandey R, Singh O. New stable numerical solutions of singular integral equations of abel type by using normalized Bernstein polynomials. Appl Math Sci 2009;3:241–55.
[27] Hu Y, Chen J, Chi S. Perturbation and stability analysis of strong form collocation with reproducing kernel approximation. Internat J Numer Methods Engrg 2011;88:157–79.
[28] Nawaz R, Khattak A, Akbar M, Ahsan S, Shah Z, Khan A. Solution of fractional?order integro?differential equations using optimal homotopy asymptotic method. J Therm Anal Calorim 2020.
[29] Oyedepo T, Taiwo O, Abubakar J, Ogunwobi Z. Numerical studies for solving fractional integro-differential equations by using least squares method and Bernstein polynomials. Fluid Mech 2016;3:1–7.
[30] Yi M, Wang L, Huang J. Legender wavelets method for the numerical solution off fractional integro-differential equations with weakly singular kernel. Appl Math Model 2016;40:3422–37.
[31] Nematia S, Sedaghat S, Mohammadic I. A fast numerical algorithm based on the second kind Chebyshev polynomials for fractional integro-differential equations with weakly singular kernels. J Comput Appl Math 2016;308:231–42.
[32] Wang1 Y, Zhu L, Wang Z. Fractional-order Euler functions for solving fractional integro-differential equations with weakly singular kernel. Adv Difference Equ 2018;254.
[33] Sahu P, Ray S. A novel Legendre wavelet Petrov–Galerkin method for fractional Volterra integro-differential equations. Comput Math Appl 2016.