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On Complete, Horizontal and Vertical Lifts From a Manifold With $f_\lambda(6,4)$ Structure to Its Cotangent Bundle

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Abstract. Manifolds with $f_\lambda(6,4)$ structure was defined and studied in the past. Later the geometry of tangent and cotangent bundles in a differentiable manifold with $f_\lambda(6,4)$ structure was studied. The aim of the present paper is to study complete, horizontal and vertical lifts from a manifold with $f_\lambda(6,4)$-structure to its cotangent bundle.

1. Introduction

The research on the properties of tensorial structure on manifolds and its extension to tangent and cotangent bundles is always gaining attraction from the researchers. Yano [12], [13], [14] introduced the idea of horizontal and vertical lifts on the tangent bundles. Kim [6] studied properties of f manifold. Dube [5], Upadhyay and Gupta [11] studied integrability conditions of $f^{2\nu+4} + f^2 = 0; f^6 = 0$ and of type (1; 1) and $F(K; -(K-2))$ - structure satisfying $F^K - F^{K-2} = 0$; $(F \neq 0; I)$. Srivastava [9], [10] studied complete lifts of (1,1) tensor field F satisfying structure $F^{\nu+1} - \lambda^2 F^{\nu-1} = 0$ and extended in M^n to cotangent bundle. Nivas and Saxena [8] studied horizontal and complete lifts from a manifold with $f_\lambda(7,-1)$ structure to its cotangent bundles. Li and Krupka [7] discussed the properties of tangent bundles. Cayir [1], [2] and [3] studied lifts of $F^{\nu+1}, \lambda^2 F^{\nu-1}$ structure.

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Let M be a differentiable manifold of class C^∞ and let ${}^C TM$ denote the cotangent bundle of M. Then ${}^C TM$ is also a differentiable manifold of class C^∞ and dimension $2n$. Throughout this paper we shall use the following notations and conventions:

(i) The map n : ${}^C\mathrm{TM} \to M$ denotes the projection map of ${}^C TM$ onto M.

(ii) Suffixes $a, b, c \ldots h, i, j \ldots$ take value 1 to n and $\bar{i} = i + n$. Suffixes $A, B, C, \ldots,$ take the value 1 to $2n$.

(iii) J_B^r (M) denote the set of tensor fields of class C^∞ and type (r,s) on M. Similarly J_B^r (${}^C TM$) denotes the set of such tensor fields in ${}^C TM$.

(iv) Vector fields in M are denoted by $X, Y, Z \ldots$ and the Lie-derivative by L_X.

(v) The Lie product of X, Y is denoted by $[X, Y]$.

If A is a point in M and n^{-1} (A) is a fibre over A. Any point $\bar{p} \in n^{-1}$ (A) is the ordered pair (A, P_A), where p is 1-form in M and 'P_A' is the value of p at A. Let U be a coordinate neighborhood in M such that $A \in U$. Then U induces a coordinate neighbourhood n^{-1} (U) in ${}^C TM$ and $\bar{p} \in n^{-1}$ (U) by [4].

2. Complete Lift of $f_\lambda(6,4)$ - Structure

Let M be an $n-$dimensional differentiable manifold of class C^∞. Suppose there exists on M, a tensor field f $(\neq 0)$ of type $(1,1)$ by [6] and [10] we have

$$f^6 - \lambda^2 f^4 = 0 \tag{2.1}$$

where λ is a complex number not equal to zero. In such a manifold M, let us put

$$l = \frac{f^4}{\lambda^2}, \; m = I - \frac{f^4}{\lambda^2} \tag{2.2}$$

where I denote the unit tensor field. Then it can be easily shown that

$$l^2 = l, \quad m^2 = m, \; l + m = I \; and \; l * m = m * l = 0 \tag{2.3}$$

Thus, the operators 'l' and 'm' when applied to the tangent space M at a point are complementary projection operators. Hence there exist complementary distributions L^* and M^* corresponding to the projection operators 'l' and 'm' respectively. If the rank of f is constant everywhere and equal to r, the dimension of L^* and M^* are r and $(n\text{-}r)$ respectively. Let us call such a structure on M as $f_\lambda(6,4)$ - structure of rank r.

Let f_i^h be component of f at A in the coordinate neighbourhood U of M. Then the complete lift f^C of f is also a tensor field of type $(1,1)$ in ${}^C TM$, where components $\bar{f}_B^A$ in $\pi^{-1}(U)$ are given by [4]

$$\overline{f}_i^h = f_i^h \ ; \ \overline{f}_{\bar{i}}^h = 0 \ ;$$
$$\overline{f}_i^{\bar{h}} = p_a \left(\frac{\partial f_h^a}{\partial x^i} - \frac{\partial f_i^a}{\partial x^h} \right); \ \overline{f}_{\bar{i}}^{\bar{h}} = f_h^i \tag{2.4}$$

where $(x^1, x^2, \ldots, x^n)$ are coordinates of A in U and p_a has components $(p_1, p_2, \ldots, p_n)$. Thus we can, write

$$f^C = \left(\overline{f}_B^A \right) = \begin{bmatrix} f_i^h & 0 \\ p_a \left(\partial i f_h^a - \partial h f_i^a \right) & f_h^i \end{bmatrix} \tag{2.5}$$

where $\partial_i = \partial / \partial x^i$.

If we put

$$\partial i f_h^a - \partial h \ f_i^a = 2\partial \ [i \ f_h^a],$$

Then we can write $\overline{f}_B^A$ as

$$f^C = \left(\overline{f}_B^A \right) = \begin{bmatrix} f_i^h & 0 \\ 2p_a \partial \ \left[i f_h^a \right] & f_h^i \end{bmatrix} \tag{2.6}$$

Thus, we have

$$\left(f^C\right)^2 = \begin{bmatrix} f_i^h & 0 \\ 2p_a \partial \ \left[i f_h^a \right] & f_h^i \end{bmatrix} \begin{bmatrix} f_j^i & 0 \\ 2p_t \partial \ \left[j f_i^t \right] & f_i^j \end{bmatrix}$$

Or

$$\left(f^C\right)^2 = \begin{bmatrix} f_i^h f_j^i & 0 \\ 2p_a f_j^i \partial \ \left[i f_h^a \right] + \ 2p_t f_h^i \partial \ \left[j f_i^t \right] & f_i^j f_h^i \end{bmatrix} \tag{2.7}$$

If we put

$$2p_a f_j^i \partial \ [i f_h^a] + \ 2p_t f_h^i \partial \ \left[j f_i^t \right] = \ L_{hj} \tag{2.8}$$

$$\left(f^C\right)^2 = \begin{bmatrix} f_i^h f_j^i & 0 \\ L_{hj} & f_i^j f_h^i \end{bmatrix} \tag{2.9}$$

Squaring again from [4] we get

$$\left(f^C\right)^4 = \begin{bmatrix} f_i^h f_j^i & 0 \\ L_{hj} & f_i^j f_h^i \end{bmatrix} \begin{bmatrix} f_k^j f_l^k & 0 \\ L_{jl} & f_k^l f_j^k \end{bmatrix}$$

Or

$$\left(f^C\right)^4 = \begin{bmatrix} f_i^h f_j^i f_k^j f_l^k & 0 \\ f_k^j f_l^k L_{hj} + \ f_i^j f_h^i L_{jl} & f_k^l f_j^k f_i^j f_h^i \end{bmatrix} \tag{2.10}$$

Thus

$$\left(f^C\right)^6 = \begin{bmatrix} f_i^h f_j^i f_k^j f_l^k & 0 \\ f_k^j f_l^k L_{hj} + \ f_i^j f_h^i L_{jl} & f_k^l f_j^k f_i^j f_h^i \end{bmatrix} \begin{bmatrix} f_m^l f_n^m & 0 \\ L_{ln} & f_m^n f_l^m \end{bmatrix} \tag{2.11}$$

Or

$$(f^C)^6 = \begin{bmatrix} f_i^h f_j^i f_k^j f_l^k f_m^l f_n^m & 0 \\ f_k^j f_l^k f_m^l f_n^m L_{hj} + f_i^j f_h^i f_m^l f_n^m L_{jl} + f_k^l f_j^k f_i^j f_h^i L_{ln} & f_m^n f_l^m f_k^l f_j^k f_i^j f_h^i \end{bmatrix}$$

Putting again

$$\begin{aligned} f_k^j f_l^k f_m^l f_n^m L_{hj} + f_i^j f_h^i f_m^l f_n^m L_{jl} + f_k^l f_j^k f_i^j f_h^i L_{ln} \\ = \lambda^2 \{ f_q^p f_n^q L_{hp} + f_r^p f_h^r L_{pn} \} \end{aligned} \tag{2.12}$$

Thus, in view of the equations (2.12) and also (2.1), the above equation (2.11) takes the form

$$(f^C)^6 = \begin{bmatrix} \lambda^2 f_p^h f_q^p f_r^q f_n^r & 0 \\ \lambda^2 \left\{ f_q^p f_n^q L_{hp} + f_r^p f_h^r L_{pn} \right\} & \lambda^2 f_r^n f_q^r f_p^q f_h^p \end{bmatrix}$$

$$= \lambda^2 \begin{bmatrix} f_p^h f_q^p f_r^q f_n^r & 0 \\ f_q^p f_n^q L_{hp} + f_r^p f_h^r L_{pn} & f_r^n f_q^r f_p^q f_h^p \end{bmatrix}$$

Or $(f^C)^6 - \lambda^2 (f^C)^4 = 0$

Hence the complete lift f^C of f also has $f_\lambda(6,4)$ - structure in the cotangent bundle ${}^C TM$. Thus, we have

Theorem 2.1. *In order that the complete lift f^C of a (1,1) tensor field f admitting $f_\lambda(6,4)$-structure in M may have the similar structure in the cotangent bundle ${}^C TM$, it is necessary and sufficient that*

$$f_k^j f_l^k f_m^l f_n^m L_{hj} + f_i^j f_h^i f_m^l f_n^m L_{jl} + f_k^l f_j^k f_i^j f_h^i L_{ln} = \lambda^2 \left\{ f_q^p f_n^q L_{hp} + f_r^p f_h^r L_{pn} \right\}$$

3. Nijenhuis Tensor of Complete Lift of f^6

The Nijenhuis tensor of *(1,1)* tensor field f on M is given by

$$N_{f,f}(X,Y) = [fX, fY] - f[fX, Y] - f[X, fY] + f^2[X,Y] \tag{3.1}$$

Also, for the complete lift of f^6, the Nijenhuis tensor is given by

$$\begin{aligned} N_{(f^6)^c,(f^6)^c}(X^c,Y^c) = [(f^6)^c X^c, (f^6)^c Y^c] - (f^6)^c [(f^6)^c X^c, Y^c] \\ - (f^6)^c [X^c, (f^6)^c Y^c] + (f^6)^c (f^6)^c [X^c, Y^c] \end{aligned} \tag{3.2}$$

In the view of the equation (2.1), the above equation takes the form

$$\begin{aligned} N_{(f^6)^c,(f^6)^c}(X^c,Y^c) = [(\lambda^2 f^4)^c X^c, (\lambda^2 f^4)^c Y^c] \\ -(\lambda^2 f^4)^c \left[(\lambda^2 f^4)^c X^c, Y^c\right] \\ -(\lambda^2 f^4)^c \left[X^c, (\lambda^2 f^4)^c Y^c\right] \\ +(\lambda^2 f^4)^c (\lambda^2 f^4)^c [X^c, Y^c] \end{aligned} \tag{3.3}$$

$$= \lambda^4\{[(f^4)^c\, X^c\, ,\, (f^4)^c Y^c] \,-(f^4)^c\left[(f^4)^c X^c,\ Y^c\right]$$
$$-(f^4)^c\left[X^c,\ (f^4)^c\ \ Y^c\right]+(f^4)^c(f^4)^c[X^c\ ,\ Y^c]\}$$

Also,

$$(f^4)^c X^c \ =\ (f^4X)^c\ +\nu(L_X f^4) \tag{3.4}$$

where (νf) has components

$$(\nu f)=\left[p_a^0\ f_i^a\right] \tag{3.5}$$

In view of the equation (3.4), the equation (3.3) takes the form of a horizontal lift of $f_\lambda(6,4)$ structure.

$$N_{\left((f^4)^C,(f^4)^C\right)}\left(X^C,\ Y^C\right)=\lambda^4\ \Big\{\Big[(f^4X)^C,\ (f^4Y)^C\Big]\ +\Big[\nu\left(L_Xf^4\right),\ (f^4Y)^C\Big]\ +\Big[\left(f^4X\right)^C,\nu\left(L_Yf^4\right)\Big]$$
$$+\left[\ \nu\left(L_Xf^4\right),\ \nu\left(L_Yf^4\right)\right]-\ \ (f^4)^C\left[(f^4X)^C,\ Y^C\right]-(f^4)^C\left[\ \nu\left(L_Xf^4\right)^C,\ Y^C\right]$$
$$-\ \ (f^4)^C\left[X^C,(f^4Y)^C\right]-\ (f^4)^C\left[X^C,\nu\left(L_Yf^4\right)\right]+\ (f^4)^C(f^4)^C[X^C,\ Y^C]\Big\} \tag{3.6}$$

Let us now suppose that

$$L_Xf^4-\ L_Yf^4=0 \tag{3.7}$$

The equation (3.6) takes the form

$$N_{\left((f^4)^C\right),(f^4)^C}\left(X^C,\ Y^C\right)=\ \lambda^4\ \left\{[(f^4X)^C,\ (f^4Y)^C]\right\}-\ \ (f^4)^C\left[(f^4)^C,\ Y^C\right] \tag{3.8}$$

$$-\ \ (f^4)^C\left[X^C,(f^4Y)^C\right]+\ (f^4)^C(f^4)^C[X^C,\ Y^C]$$

Suppose further that the (1,1) tensor field f satisfies

$$f^4=\ \ \lambda^2 I \tag{3.9}$$

Then in the view of the equation (3.8), the equation (3.7) takes the form of

$$N_{\left((f^4)^C\right),(f^4)^C}\left(X^C,\ Y^C\right)=\ \lambda^8\ \left\{\left[X^C,\ Y^C\right]-\ \left[X^C,\ Y^C\right]-\ \left[X^C,\ Y^C\right]+\ \left[X^C,\ Y^C\right]\right\}=0.$$

Hence, we have

Theorem 3.1. *The Nijenhuis tensor of the complete lift of f^6 vanishes if the Lie – derivatives of the tensor field f^4 with respect to X and Y are both zero and the tensor field f^2 acts as GF- structure operator on M.*

4. Horizontal Lift of $f_\lambda(6,4)$- Structure

Let *f*, *g* be the tensor fields of type (1, 1) of manifold M. If f^H be the horizontal lift of *f*, we have by [4] and [14]

$$f^H g^H + g^H f^H = (fg + gf)^H \tag{4.1}$$

Equating f and g, we get

$$(f^H)^2 = (f^2)^H \tag{4.2}$$

Squaring equation (4.2) on both sides we get,

$$(f^H)^4 = (f^4)^H \tag{4.3}$$

Taking cube of (4.2) and using (4.2) itself and (4.3) we get,

$$(f^H)^6 = (f^6)^H \tag{4.4}$$

Since f gives $f_\lambda(6,4)$ – structure on M, so

$$f^6 - \lambda^2 f^4 = 0$$

Taking horizontal lift in the above equation we get,

$$(f^6)^H - \lambda^2 (f^4)^H = 0 \tag{4.5}$$

In view of the equation (4.3) and (4.4), the above equation (4.5) takes the form

$$(f^6)^H - \lambda^2 (f^H)^4 = 0$$

Thus, we have the following theorem:

Theorem 4.1. *Let f be the tensor field of type (1, 1) admitting $f_\lambda(6,4)$ structure in M. Then the horizontal lift f^H of f also admits the similar structure in the cotangent bundle c_{TM}.*

5. Vertical Lift of $f_\lambda(6,4)$- Structure

Let *f*, *g* be the tensor fields of type (1, 1) of manifold M. If f^V be the vertical lift of *f*, we have

$$f^V g^V + g^V f^V = (fg + gf)^V \tag{5.1}$$

Equating f and g, we get

$$(f^V)^2 = (f^2)^V \tag{5.2}$$

Squaring equation (5.2) on both sides we get,

$$(f^V)^4 = (f^4)^V \tag{5.3}$$

Taking cube of (5.2) and using (5.2) itself and (5.3) we get,

$$(f^V)^6 = (f^6)^V \tag{5.4}$$

Since f gives $f_\lambda(6,4)$ – structure on M, so

$$f^6 - \lambda^2 f^4 = 0$$

Taking vertical lift in the above equation we get,

$$(f^6)^V - \lambda^2 (f^4)^V = 0 \tag{5.5}$$

In view of the equation (5.3) and (5.4), the above equation (5.5) takes the form

$$(f^6)^V - \lambda^2 (f^V)^4 = 0$$

Thus, we have the following theorem:

Theorem 5.1. *Let f be the tensor field of type (1, 1) admitting $f_\lambda(6,4)$ structure in M. Then the vertical lift f^V of f also admits the similar structure in the cotangent bundle c_{TM}.*

6. Conclusion

In this research, $f_\lambda(6,4)$ structure has been defined on an n-dimensional differentiable manifold of class C^∞. Further properties of complete, horizontal and vertical lifts of $f_\lambda(6,4)$ structure are defined on its cotangent bundle. The necessary and sufficient conditions for cotangent bundles to have the properties of M in complete, horizontal and vertical lifts are also discussed. Properties of Nijenhuis tensor of complete lift of f^6 is also a part of this paper.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] H. Cayir, Tachibana and Vishnevskii Operators Applied to X^V and X^H in Almost Paracontact Structure on Tangent Bundle $T(M)$, New Trends Math. Sci. 4 (2016), 105-115. https://doi.org/10.20852/ntmsci.2016318821.

[2] H. Cayir, Lie Derivatives of Almost Contact Structure and Almost Paracontact Structure With Respect To X^v and X^H on Tangent Bundle $T(M)$, Proc. Inst. Math. Mech. 42 (2016), 38-49.

[3] H. Cayir, Some Notes on Lifts of the $((\nu+1), \lambda^2(\nu-1))$ - Structure on Cotangent and Tangent Bundle, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. 70 (2021), 241–264. https://doi.org/10.31801/cfsuasmas.712861.

[4] L.S. Das, R. Nivas, V.N. Pathak, On Horizontal and Complete Lifts From a Manifold With $f_\lambda(7;1)$-Structure to Its Cotangent Bundle, Int. J. Math. Math. Sci. 8 (2005), 1291-1297. https://doi.org/10.1155/IJMMS.2005.1291.

[5] K.K. Dube, On a Differentiable Structure Satisfying $f^{2v+4} + f^2 = 0; f^6 = 0$ of Type (1; 1), Nepali Math. Sci. Rep. 17 (1998), 99-102.

[6] J.B. Kim, Notes on f-Manifold, Tensor N-S, 29 (1975), 299-302.

[7] T. Li, D. Krupka, The Geometry of Tangent Bundles: Canonical Vector Fields, Geometry. 2013 (2013), 364301. https://doi.org/10.1155/2013/364301.

[8] R. Nivas, M. Saxena, On Complete and Horizontal Lifts From a Manifold With HSU-(4; 2) Structure to Its Cotangent Bundle, Nepali Math. Sci. Rep. 23 (2004), 35-41.

[9] S.K. Srivastava, R. Nivas, On Horizontal & Complete Lifts From a Manifold With $f_\lambda(7,-1)$ Structure to Its Cotangent Bundle, J. Tensor Soc. India, 14 (1996), 42-48.

[10] S.K. Srivastava, On the Complete Lifts of (1, 1) Tensor Field F Satisfying Structure $F^{\nu+1} - \lambda^2 F^{\nu-1} = 0$, Nepali Math. Sci. Rep. 21 (2003), 89-99.

[11] M.D. Upadhyay, V.C. Gupta, Integrability Conditions of a $F(K; -(K-2))$ - Structure Satisfying $F^K - F^{K-2} = 0$; $(F \neq 0; I)$, Rev. Univ. Nac. Tucuman, 20 (1976), 31-44.

[12] K. Yano, On a Structure Defined by a Tensor Field f of Type (1,1) Satisfying $f^3 + f = 0$, Tensor N. S. 14 (1963), 99-109.

[13] K. Yano, C.S. Houh, B.Y. Chen, Structures Defined by a Tensor Field ϕ of Type (1,1) Satisfying $\phi^4 \pm \phi^2 = 0$, Tensor N. S. 23 (1972), 81-87.

[14] K. Yano, S. Ishihara, Tangent and Cotangent Bundles, Marcel Dekker Inc., New York, (1973).