2-Uniform covering groups of elementary abelian 2-groups

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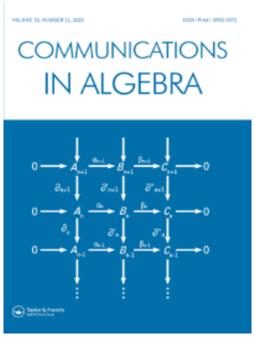

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2-Uniform covering groups of elementary abelian 2-groups

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ABSTRACT

This article is concerned with the classification of Schur covering groups of the elementary abelian group of order 2^n, up to isomorphism. We consider those covering groups possessing a generating set of n elements having only two distinct squares. We show that such groups may be represented by 2-vertex-colored and 2-edge-colored graphs of order n. We show that in most cases, the isomorphism type of the group is determined by that of the 2-colored graph, and we analyze the exceptions.

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1. Introduction

For a finite group G, a *Schur cover* or *covering group* or *stem cover* of G is a finite group H with a normal subgroup $N \subseteq Z(H) \cap H'$ with $H/N \cong G$, that has maximal order amongst all groups with this property. A pair of groups (H, N) with $N \subseteq Z(H) \cap H'$ and $H/N \cong G$ is referred to as a *stem extension* of G. Thus a covering group is a stem extension of maximal order. A group may have multiple non-isomorphic covering groups, but in all cases the normal subgroup N is isomorphic to the Schur Multiplier $M(G)$ of G. We refer to Chapter 7 of [4] for an account of the general theory of covering groups and their role in the study of projective representations.

The theme of this article is the classification, up to isomorphism, of covering groups of elementary abelian 2-groups. For a prime p and positive integer n, the elementary abelian p-group of order p^n is the direct product of n copies of the cyclic group C_p of order p. Written additively, it is the vector space of dimension n over the field $\mathbb{F}_p$ of p elements. Elementary abelian groups possess a particular abundance of distinct covering groups.

If (H, N) is a stem extension of an abelian group G, then $N = H'$ and H is either abelian or nilpotent of class 2. Then the following commutator identities are satisfied for all elements x, y, z of H, where $[x, y]$ denotes the element $xyx^{-1}y^{-1}$.

$$[x,z][y,z] = [xy,z], \text{ and } [x,y][x,z] = [x,yz]. \tag{1}$$

Consequently, for elements x, y of H and any positive integer t, the commutator $[x, y]$ satisfies $[x, y]^t = [x^t, y] = [x, y^t]$. In particular, if either x^t or y^t is central in H, then $[x, y]^t = \text{id}$ in N. Since N is abelian and generated by commutators, it follows that the exponent of N divides that of G. In particular, if G is an elementary abelian p-group, then either N is trivial or it is also elementary abelian of exponent p. If $\{x_1, \dots, x_n\}$ is a set of elements of H for which H/N is generated by the x_iN, then $\{x_1, \dots, x_n\}$ generates H, and N is generated by the $\binom{n}{2}$ simple commutators $[x_i, x_j]_{i<j}$, each of which is either trivial

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or has order p. The maximum possible order of N is $p^{\binom{n}{2}}$, and this order occurs when the $[x_i, x_j]$ are independent. When this occurs, H is a *covering group* of C_p^n. The full structure of H, in terms of the elements of the prescribed generating set $\{x_1, \ldots, x_n\}$, is determined by the expression for each of the elements x_i^p as a product of powers of the simple commutators $[x_i, x_j]_{i<j}$.

A covering group of C_p^n may be constructed by specifying a set of generators $x_1, \ldots, x_n$ and freely choosing the integers a_{ijk} from $\{0, \ldots, p-1\}$ in the n expressions

$$x_k^p = \prod_{1 \le i < j \le n} [x_i, x_j]^{a_{ijk}}. \tag{2}$$

This point will be discussed in more detail in Section 2. The number of choices available for the collection of indices $\{a_{ijk}\}$ is $p^{n\binom{n}{2}}$. While a superficial inspection shows that many different choices yield isomorphic covering groups, it is also clear that many non-isomorphic examples may occur as the value of n increases. In [5], Ursula Martin Webb investigates the number $A(p, n)$ of all isomorphism types of covering groups of C_p^n for odd p and shows that it is bounded below by

$$\frac{p^{n\binom{n}{2}}}{|\mathrm{GL}(n,p)|}\left(p^{-3n^2/2+9n/2-4}(p^n-1)(p+p^{n-1}-1)(p-1)+1\right).$$

This result alone shows that the elementary abelian group of order 81 has at least 12555 distinct covering groups. The term $p^{n\binom{n}{2}}$ that appears in the numerator of the above expression is the number of choices for the coefficients a_{ijk} in the expression (2).

Our aim in this article is not to attempt an enumeration of all isomorphism types, but to consider how isomorphic groups can be recognized on the basis of descriptions of the form in (2), possibly for different choices of distinguished generating sets. Our focus is on the case of 2-groups, for which the analysis is markedly different from that of odd primes, mostly because of the following observation, which is a direct consequence of (1).

Lemma 1.1. *Let G be a covering group of an elementary abelian p-group, for an odd prime p. Then the pth power map on G, defined by $x \to x^p$ for $x \in G$, is a group homomorphism.*

Proof. Let $x, y \in G$. Then $yx = xy[x, y]$ and, since $[x, y] \in Z(G)$ it follows that

$$(xy)^p = x^p y^p [x, y]^{\frac{p(p-1)}{2}}.$$

Since $p-1$ is even, the integer $\frac{p(p-1)}{2}$ is a multiple of p, and since G' has exponent p it follows that $(xy)^p = x^p y^p$. □

For a covering group G of an elementary abelian p-group of odd order p^n, Lemma 1.1 may be interpreted as the statement that the pth power map is a linear transformation from G/G', which is a vector space of dimension n over $\mathbb{F}_p$, to G', which is a vector space of dimension $\binom{n}{2}$ over $\mathbb{F}_p$. Since this mapping determines the group G up to isomorphism, the problem of distinguishing and classifying covering groups may be regarded as a problem of linear algebra. We define the *rank* of the covering group G to be the rank of its pth power mapping as a linear transformation, and note that isomorphic covering groups have the same rank. The rank of G is k if the elementary abelian subgroup of G consisting of all pth powers has order p^k. For odd p, C_p^n has one covering group of exponent p and rank 0. Covering groups of rank 1 are those in which the pth powers comprise a single cyclic group of order p. They are investigated in [1], where it is shown that the number of their isomorphism types is $n-1$.

Since $(xy)^2 = x^2y^2[x, y]$ for all elements x, y of any group G, the squaring map in a covering group of a non-cyclic elementary abelian 2-group is never a homomorphism. The set of squares in such a group is not a subgroup, and so the concept of rank, at least in terms of linear transformations, does not translate directly to the case of 2-groups. Nevertheless, by considering the least number of distinct elements that may occur as the squares of the elements of a generating set, we propose an invariant for 2-groups that may be regarded as an analogue of rank.

For odd p, a covering group of rank 1 of C_p^n is one that is generated by n elements all having the same pth power. This version of the definition may also apply to 2-groups, and we say that a covering group of C_2^n is *uniform* if it possesses a generating set consisting of n elements all having the same square. Included in this designation is the unique covering group that is generated by n involutions. A detailed investigation of uniform covering groups is presented in [3]. It is shown there that the number of isomorphism types of uniform covering groups of C_2^n is equal to the number of isomorphism types of simple undirected graphs on n unlabeled vertices. This number generally greatly exceeds the corresponding number $n-1$ for the case of odd p and rank 1, reflecting the fact that the classification problem is one of combinatorics rather than linear algebra. The goal of this article is to extend the investigation to the class of *2-uniform* covering groups, which are those non-uniform covering groups that possess a generating set whose elements have just two distinct squares. We begin with some further background information from [3], about the uniform property and its connection to graphs.

2. Uniform covering groups of elementary abelian 2-groups

In this section, we discuss the graph representation of central elements of covering groups of elementary abelian 2-groups.

Definition 2.1. For a positive integer n, a covering group of the elementary abelian 2-group C_2^n is a group G of order $2^{n+\binom{n}{2}}$ with the following properties:

- G has a generating set $\{x_1, \dots, x_n\}$ with n elements.
- The commutator subgroup of G is equal to the center of G, and is an elementary abelian group of order $2^{\binom{n}{2}}$, generated by the $\binom{n}{2}$ simple commutators $[x_i, x_j]$, with $i < j$.
- $G/Z(G) = G/G'$ is elementary abelian of order 2^n, generated by the cosets of G' represented by $x_1, \dots, x_n$.

We now let G be a covering group of C_2^n. We will refer to any minimal generating set of G as a *basis* of G. We say that a subset of G is *independent* if its elements represent linearly independent elements of G/G', regarded as a vector space over $\mathbb{F}_2$. Thus a basis is a maximal independent set. Since the commutator subgroup of G has exponent 2, the commutators $[x, y]$ and $[y, x]$ coincide for all pairs of elements x and y of G, and we may consider the element $[x, y]$ to be determined by the unordered pair $\{x, y\}$. We now let $\mathcal{B} = \{x_1, \dots, x_n\}$ be a basis of G, and introduce a set V of n vertices labeled by the elements of $\mathcal{B}$. For $1 \le i < j \le n$, the *basic simple commutator* $[x_i, x_j]$ is represented by the graph on V whose only edge comprises the two vertices labeled by x_i and x_j. Every element of G' has an expression as a product of distinct basic simple commutators, which is unique up to order. Thus each $c \in G'$ is represented by the graph on V whose edges correspond to the pairs of elements of $\mathcal{B}$ whose commutators occur in c. The choice of a basis of G determines a bijective correspondence between G' and the collection of all graphs on n labeled vertices.

Let F be a free group of rank n, with generators $X_1, \dots, X_n$. and let G be a covering group of C_2^n with basis $\{x_1, \dots, x_n\}$. Then there is an epimorphism $\phi : F \to G$ with $\phi(X_i) = x_i$ for each i. Since G has exponent 4, G' has exponent 2, and all commutators and squares are central in G, the kernel R of ϕ contains the subgroup H of F generated by all elements of the forms

$$X^4,\ [X^2, Y],\ [X, Y]^2, [[X, Y], Z],\ \text{for } X, Y, Z \in F.$$

We write $\bar{X}_i$ for the element of F/H represented by X_i. The center of the group F/H is elementary abelian of order $2^{\frac{n(n+1)}{2}}$, generated by $\bar{X}_1^2, \dots, \bar{X}_n{}^2$ and the $\binom{n}{2}$ simple commutators $[\bar{X}_i, \bar{X}_j]_{i<j}$. See [2] for a discussion of this point. Since the center of F/H strictly contains the commutator subgroup, F/H is not a covering group of C_2^n. However, every covering group of C_2^n may be realized as a quotient of

F/H, modulo a subgroup C of order 2^n that is a complement of $(F/H)'$ in $Z(F/H)$. Such a subgroup is elementary abelian, generated by elements of the form $X_1^2c_1, \dots X_n^2c_n$, where each c_i belongs to $(F/H)'$ and has a unique expression as a product of the $[\bar{X}_i, \bar{X}_j]$. If $c_i = \theta_i(\bar{X}_i, \dots, \bar{X}_n)$ and $G = (F/H)/C$, then $x_i^2 = \theta_i(x_1, \dots, x_n)$ in G. Choosing a complement C of $(F/H)'$ in $Z(F/H)$ amounts to designating the square of each of the generators $x_1, \dots, x_n$ of G as a product of the basic simple commutators $[x_i, x_j]$. This can be done freely and independently for each x_i, with different choices corresponding to different choices of C. Different choices for the squaring map on generators may lead to isomorphic covering groups, and determining when this occurs is a difficult problem in general.

Example 2.2. In the case $n = 2$, a covering group of $C_2 \times C_2$ has generators x_1, x_2 and commutator subgroup of order 2, generated by $[x_1, x_2]$. This information, along with the expressions for x_1^2 and x_2^2 as elements of G', determines the group. We have four choices for the pair (x_1^2, x_2^2).

1. $x_1^2 = x_2^2 = \text{id}$. In this case G is generated by two involutions, and their product x_1x_2 satisfies $(x_1x_2)^2 = [x_1, x_2]$. For this choice, G is isomorphic to D_8, the dihedral group of order 8.
2. $x_1^2 = \text{id}$, $x_2^2 = [x_1, x_2]$. In this case $(x_1x_2)^2 = \text{id}$ and again G is dihedral of order 8, generated by two involutions.
3. $x_1^2 = [x_1, x_2], x_2^2 = \text{id}$. This is equivalent to 2. above, again $G \cong D_8$.
4. $x_1^2 = x_2^2 = [x_1, x_2]$. In this case the elements x_1, x_2 and x_1x_2 all have order 4, and G is isomorphic to the quaternion group of order 8.

Thus both non-abelian groups of order 8 are covering groups of $C_2 \times C_2$.

Definition 2.3. A covering group G of C_2^n is *uniform* if it has a basis consisting of n elements all having the same square. A basis with this property is called a *uniform basis*.

To designate a uniform covering group from the starting point of a uniform basis $\{x_1, \dots, x_n\}$, we need only select a subcollection of the $\binom{n}{2}$ commutators $[x_i, x_j]_{i<j}$, whose product is the common square of the basis elements. This amounts to selecting a graph on n vertices, labeled by the basis elements. For a uniform covering group G with uniform basis $\mathcal{B}$, we define $\Gamma_{\mathcal{B}}(G)$ to be the graph with vertices labeled by the elements of $\mathcal{B}$, in which the vertices labeled by x_i and x_j are adjacent if and only if $[x_i, x_j]$ appears in the expression for the common square of all elements of $\mathcal{B}$ as a product of the basic simple commutators.

It is shown in [3] that the isomorphism type of the graph determined by a uniform covering group of C_2^n does not depend on the choice of uniform basis, and that in most cases there is a unique uniform basis, up to the cosets of G' to which the generators belong. The exception to this is the case where the group is represented by a clique on an even number of vertices, and in this case distinct uniform bases correspond to isomorphic graphs. The following statement is Theorem 2.11 of [3].

Theorem 2.4. *For a positive integer n, the isomorphism types of uniform covering groups of C_2^n are in bijective correspondence with the isomorphism classes of simple undirected graphs on n unlabeled vertices.*

Example 2.5. Both the dihedral group D_8 of order 8 and the quaternion group Q_8 of order 8 are uniform covering groups of $C_2 \times C_2$. The dihedral group is generated by a pair of involutions, corresponding to the null graph on two vertices, and the quaternion group is generated by a pair of elements x_1 and x_2, with $x_1^2 = x_2^2 = [x_1, x_2]$. This group is represented by the complete graph on two vertices. It is only in the case $n = 2$ that all covering groups of C_2^n are uniform.

Subject to the choice of a basis $\mathcal{B}$ for a covering group G of C_2^n, any element of G' may be described, as outlined above, by a graph on a set of n vertices labeled by the elements of $\mathcal{B}$. The distinguishing feature of uniform covering groups is that a single such graph is sufficient to fully specify the group. Our theme in this article is to explore the case of covering groups that are not uniform but possess a basis whose elements have only two distinct squares. Such groups will be called 2-*uniform*, and they can be described using graphs with a 2-coloring of both their vertex and edge sets.

For an element c of G', the graph of c with respect to $\mathcal{B}$ is denoted by $\Gamma_{\mathcal{B}}(c)$. Its vertices are labeled by the elements of $\mathcal{B}$, and its edges are those pairs of basis elements that appear as commutators in the unique expression for c as a product of basic simple commutators from $\mathcal{B}$. One may consider the relationships between the graphs that represent c with respect to different bases of G. The case of a pair of bases that differ only in one or two elements will be of particular interest, and we conclude this section by noting the graph transformations that correspond to basis changes of this nature. If $\mathcal{B}$ and $\mathcal{B}'$ are two bases of G that differs in either exactly one or exactly two elements, we assume that $\Gamma_{\mathcal{B}}(c)$ and $\Gamma_{\mathcal{B}'}(c)$ have the same vertex set, with the relevant vertex or pair of vertices relabeled in the transition from one graph to the other.

If the element c is a nonidentity commutator in G, then $c = [p, q]$ for some $p, q \in G$. Since c depends only on the cosets pG' and qG', we may assume that each of p and q are products of elements of $\mathcal{B}$. Let P and Q respectively denote the sets of vertices of $\Gamma_{\mathcal{B}}(c)$ that represent the basis elements that occur in p and q. Expanding the expression $[p, q]$ in terms of the basis elements, we observe that the edges of $\Gamma_{\mathcal{B}}(c)$ and their incident vertices comprise a complete tripartite graph with parts $P\backslash Q$, $Q\backslash P$ and $P \cap Q$, or a complete bipartite graph if one of these three sets is empty. It follows that a graph represents a simple commutator (i.e. an element of G' of the form $[p, q]$) if and only if it has a connected component that is complete tripartite or complete bipartite, with remaining vertices isolated. This situation will arise frequently in our analysis, so we introduce the following notation for the set of edges that represents the commutator of a pair of elements from specified cosets of G' in G.

Definition 2.6. For sets of vertices P and Q, we denote by $E(P, Q)$ the set of edges of the complete tripartite (or bipartite or null) graph whose parts are $P\backslash Q$, $Q\backslash P$ and $Q \cap P$.

In general, we write $E(\Gamma)$ for the edge set of a graph Γ. For a pair of sets A and B, $A\triangle B$ denotes the symmetric difference of A and B.

Theorem 2.7. *Suppose that $\mathcal{B}$ and $\mathcal{B}' = (\mathcal{B}\backslash\{x\}) \cup \{y\}$ are bases of G, and let $c \in G'$. Let v be the vertex that represents x in $\Gamma_{\mathcal{B}}(c)$ and y in $\Gamma_{\mathcal{B}'}(c)$. Let P be the set of neighbors of v in $\Gamma_{\mathcal{B}}(c)$, and let Q be the set of vertices representing elements of $\mathcal{B}\backslash\{x\}$ that occur in the expression for y as a product of elements of $\mathcal{B}$ (modulo G'). Then*

$$E(\Gamma_{\mathcal{B}'}(c)) = E(\Gamma_{\mathcal{B}}(c))\triangle E(P, Q).$$

Proof. Let q and p respectively denote the products (in some specified order) of the elements of $\mathcal{B}$ represented by the vertices of P and of Q. Then

$$c = [x, p]c' = [yq, p]c' = [q, p][y, p]c',$$

where c' is a product of simple commutators involving the elements of $\mathcal{B} \cap \mathcal{B}'$. Since c' is represented by the same set of edges in both graphs, and the edges that represent $[y, p]$ with respect to $\mathcal{B}'$ coincide with those that represent $[x, p]$ with respect to $\mathcal{B}$, it follows that the graph $\Gamma_{\mathcal{B}'}(c)$ is obtained from $\Gamma_{\mathcal{B}}(c)$ by switching the status of all edges that represent commutators that occur in the expansion of $[q, p]$ in terms of elements of $\mathcal{B}\backslash\{x\}$. These edges are exactly those of the set $E(P, Q)$. □

If the sets P and Q coincide in the situation of Theorem 2.7, then the graphs $\Gamma_{\mathcal{B}}(c)$ and $\Gamma_{\mathcal{B}'}(c)$ differ only in the label on the vertex v, which represents x in $\Gamma_{\mathcal{B}}(c)$ and represents y in $\Gamma'_{\mathcal{B}}(c)$. In particular, the two are isomorphic, via the unique bijection between their vertex sets that preserves the $n - 1$ labels that are common to both. We note the following special case of this situation, which will arise in our analysis.

Corollary 2.8. *Let c be an element of G' whose graph with respect to the basis $\mathcal{B}$ consists of a clique on $k \geq 2$ vertices, with any remaining vertices isolated. Let x be the product in G, in some order, of those basis elements $x_1, \ldots, x_k$ that are represented by non-isolated vertices. Let $\mathcal{B}'$ be a basis obtained from $\mathcal{B}$ by replacing some $x_i \in \{x_1, \ldots, x_k\}$ with x. Then the graphs $\Gamma_{\mathcal{B}}(c)$ and $\Gamma_{\mathcal{B}'}(c)$ are isomorphic, via the unique bijection that preserves the labels of the $n-1$ vertices representing elements common to both bases.*

We now consider the relationship between $\Gamma_{\mathcal{B}}(c)$ and $\Gamma_{\mathcal{B}''}(c)$, where $c \in G'$ and the basis $\mathcal{B}''$ is obtained from $\mathcal{B}$ by replacing two elements x_1 and x_2 with y_1 and y_2. Through two applications of Theorem 2.7, we describe the relationship between the edge sets of $\Gamma_{\mathcal{B}}(c)$ and $\Gamma_{\mathcal{B}''}(c)$. Since $\mathcal{B}$ and $\mathcal{B}''$ are both generating sets of G, we may assume that the expression for y_1 as a product of elements of $\mathcal{B}$ (modulo G') involves x_1 but not x_2, and that the corresponding expression for y_2 involves x_2. We write P_1 and P_2 respectively for the sets of neighbors of the vertices representing x_1 and x_2 in $\Gamma_{\mathcal{B}}(c)$. We write Q_1 and Q_2 for the respective sets of vertices representing elements of $\mathcal{B}\backslash\{x_1\}$ and $\mathcal{B}\backslash\{x_2\}$ that appear in the expressions for y_1 and y_2 as products of elements of $\mathcal{B}$.

We write $\mathcal{B}'$ for the basis of G that results from replacing x_1 with y_1 in $\mathcal{B}$. From a direct application of Theorem 2.7,

$$E(\Gamma_{\mathcal{B}'}(c)) = E(\Gamma_{\mathcal{B}}(c)) \triangle E(P_1, Q_1).$$

We now write P_2' for the set of neighbors of the vertex representing x_2 in $\Gamma_{\mathcal{B}'}(c)$, and Q_2' for the set of vertices representing elements of $\mathcal{B}'\backslash\{x_2\}$ that occur in the expression for y_2 as a product of elements of the basis $\mathcal{B}'$. By applying Theorem 2.7 again, we may describe the edge set of $\Gamma_{\mathcal{B}''}(c)$ in terms of the sets P_1, Q_1, P_2' and Q_2'. To describe it in terms of the original data pertaining to $\mathcal{B}$, we need to consider how P_2' and Q_2' depend on P_1, P_2, Q_1, Q_2 and the edges of $\Gamma_{\mathcal{B}}(c)$.

If the commutator $[x_1, x_2]$ occurs in the description of c in terms of simple commutators involving elements of $\mathcal{B}$, then the vertex representing x_2 belongs to $P_1\backslash Q_1$, and $P_2' = P_2 \triangle Q_1$. Otherwise $P_2' = P_2$.

If x_1 is involved in the expression for y_2 as a product of elements of $\mathcal{B}$, then the vertex representing x_1 belongs to Q_2, and $Q_2' = Q_2 \triangle Q_1$. Otherwise $Q_2' = Q_2$.

The following theorem summarizes the possible relationships between the graphs $\Gamma_{\mathcal{B}}(c)$ and $\Gamma_{\mathcal{B}''}(c)$.

Theorem 2.9. *The edge set of $\Gamma_{\mathcal{B}''}(c)$ depends on c and y_2 as follows:*

1. *If the vertices representing x_1 and x_2 are not adjacent in $\Gamma_{\mathcal{B}}(c)$, and the expression for y_2 as a product of elements of $\mathcal{B}$ does not include x_1, then*

$$E(\Gamma_{\mathcal{B}''}(c)) = E(\Gamma_{\mathcal{B}}(c)) \triangle E(P_1, Q_1) \triangle E(P_2, Q_2).$$

2. *If the vertices representing x_1 and x_2 are adjacent in $\Gamma_{\mathcal{B}}(c)$, and the expression for y_2 as a product of elements of $\mathcal{B}$ does not include x_1, then*

$$E(\Gamma_{\mathcal{B}''}(c)) = E(\Gamma_{\mathcal{B}}(c)) \triangle E(P_1, Q_1) \triangle E(P_2 \triangle Q_1, Q_2).$$

3. *If the vertices representing x_1 and x_2 are not adjacent in $\Gamma_{\mathcal{B}}(c)$, and the expression for y_2 as a product of elements of $\mathcal{B}$ includes x_1, then*

$$E(\Gamma_{\mathcal{B}''}(c)) = E(\Gamma_{\mathcal{B}}(c)) \triangle E(P_1, Q_1) \triangle E(P_2, Q_2 \triangle Q_1).$$

4. *If the vertices representing x_1 and x_2 are adjacent in $\Gamma_{\mathcal{B}}(c)$, and the expression for y_2 as a product of elements of $\mathcal{B}$ includes x_1, then*

$$E(\Gamma_{\mathcal{B}''}(c)) = E(\Gamma_{\mathcal{B}}(c)) \triangle E(P_1, Q_1) \triangle E(P_2 \triangle Q_1, Q_2 \triangle Q_1).$$

3. 2-Uniform covering groups and 2-uniform graphs

In this section, we discuss an extension of the graph representation of uniform covering groups, to the case of covering groups possessing generating sets whose elements have two distinct squares.

Definition 3.1. A covering group G of C_2^n is *2-uniform* if it is not uniform, and it has a basis $\mathcal{B}$ with the property that

$$|\{x^2 : x \in \mathcal{B}\}| = 2.$$

We refer to a basis of the type described in Definition 3.1 as a *2-square basis* of G. Any covering group that possesses a 2-square basis is either 2-uniform or uniform. We may use a 2-square basis to associate a graph to G, by extending the graph interpretation of a uniform basis as defined in Section 1. We use vertex colors to distinguish the elements of a 2-square basis according to their two distinct squares, and corresponding edge-colors to distinguish their respective squares. By a 2-colored graph, we mean a loopless undirected graph in which every vertex is colored either blue or red, and every edge is colored either blue or red. A pair of vertices may be adjacent via both a blue edge and a red edge, but multiple edges of the same color cannot occur. We say that two 2-colored graphs are *isomorphic* if there is a bijection between their vertex sets that preserves adjacency and non-adjacency, and either preserves the colors of both vertices and edges, or switches the colors of all vertices and all edges.

Let $\mathcal{B} = \{x_1, \ldots, x_k, y_{k+1}, \ldots, y_n\}$ be a 2-square basis of a covering group G of C_2^n, where $x_i^2 = r$ for $i \leq k$, $y_j^2 = s$ for $j > k$, and r and s are distinct elements of G'. We define the 2-colored graph of G with respect to the basis $\mathcal{B}$, denoted $\Gamma_{\mathcal{B}}(G)$, as follows.

- The vertex set of $\Gamma_{\mathcal{B}}(G)$ consists of k blue vertices, corresponding to the basis elements $x_1, \ldots, x_k$, and $n-k$ red vertices, corresponding to the basis elements $y_{k+1}, \ldots, y_n$;
- The blue edges of $\Gamma_{\mathcal{B}}(G)$ comprise the edge set of the graph $\Gamma_{\mathcal{B}}(r)$.
- The red edges of $\Gamma_{\mathcal{B}}(G)$ comprise the edge set of the graph $\Gamma_{\mathcal{B}}(s)$.

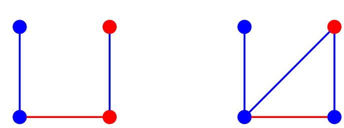
On the other hand, if Γ is a 2-colored graph, we may associate to Γ a covering group with a generator for each vertex of Γ, in which the square of each of the generators corresponding to blue vertices is the element of G' represented by the blue edges, and the square of each of the generators corresponding to red vertices is the element of G' represented by the red edges.

A 2-uniform group may have multiple 2-square bases, and may be represented by non-isomorphic 2-colored graphs, as the following example shows.

Example 3.2. Let G be the 2-uniform covering group of C_2^4 with 2-square basis $\{x_1, x_2, y_3, y_4\}$, where $x_1^2 = x_2^2 = [x_1, x_2][y_3, y_4]$, and $y_3^2 = y_4^2 = [x_1, y_3]$. Then $(x_1y_3)^2 = x_1^2y_3^2[x_1, y_3] = x_1^2$. It follows that $\{x_1, x_2, x_1y_3, y_4\}$ is another 2-square basis of G, in which

$$x_1^2 = x_2^2 = (x_1y_3)^2 = [x_1, x_2][x_1y_3, y_4][x_1, y_4],$$

and $y_4^2 = [x_1, x_1y_3]$. Thus the following nonisomorphic 2-colored graphs both represent this 2-uniform covering group G of C_2^4.



Example 3.2 shows that, even for bases consisting of elements with the same pair of squares, some variation is possible in the numbers of blue and red vertices in the corresponding graphs. This difficulty will be resolved by refining the concept of a 2-square basis to that of a *2-uniform basis*, which is one which maximizes the number of elements having a single square.

Definition 3.3. For any covering group G of C_2^n, the *uniform rank* of G, denoted $\rho(G)$, is the maximum k with the property that k independent elements of G have the same square. The *uniform corank* of G is defined as $n - \rho(G)$.

In a 2-uniform covering group of C_2^n, the uniform rank is at least $\lfloor \frac{n}{2} \rfloor$ and at most $n-1$. The uniform rank is at least equal to the uniform corank.

Definition 3.4. Let G be a 2-uniform covering group of C_2^n. A *2-uniform basis* of G is a generating set $\{x_1, \dots, x_n\}$ with the following properties:

- $x_1, \dots, x_k$ have the same square r.
- $x_{k+1}, \dots, x_n$ have the same square s, where $s \neq r$.
- k is the uniform rank of G.

We now establish that every 2-uniform covering group of an elementary abelian 2-grouo possesses a 2-uniform basis. Let G be a 2-uniform covering group of C_2^n, with uniform rank k. Let $\mathcal{B}$ be a 2-square basis of G, consisting of elements with two distinct squares r and s. If either r or s is the square of k distinct elements of $\mathcal{B}$, then $\mathcal{B}$ is a 2-uniform basis of G. Otherwise, we consider whether $\mathcal{B}$ can be adjusted to a 2-uniform basis, by the addition of further elements with one of the squares r and s, and the omission of some with the other. Such an adjustment requires that either r or s is the square of k independent elements of G. We will prove that this condition holds for every 2-square basis if $n \geq 7$, as a consequence of Theorem 3.5. The existence of 2-uniform bases in the remaining cases with $n \leq 6$ will be considered separately.

Before stating Theorem 3.5, which is one of the main technical ingredients of this work, we introduce some notation that is used in its proof. If X is a subset of a covering group G of C_2^n, we write $C(X)$ for the element of G' that is given by the product of the commutators $[x, y]$, over all unordered pairs $\{x, y\}$ of distinct elements of X. If $X = \{x_1, x_2, \dots, x_t\}$, we may write $C(x_1, \dots, x_t)$ for $C(X)$. If the elements of X are independent in G and are included in a basis $\mathcal{B}$, then $\Gamma_{\mathcal{B}}(C(X))$ consists of a clique on those vertices representing the elements of X, with remaining vertices isolated.

Theorem 3.5. *Let G be a 2-uniform covering group of C_2^n, where $n \geq 4$, and let $\mathcal{B} = \{x_1, \dots, x_k, y_{k+1}, \dots, y_n\}$ be a generating set of G, where $x_i^2 = r$ for $i = 1, \dots, k$, and $y_j^2 = s$ for $j = k+1, \dots, n$, and where $k \geq n - k$. Then no element of $G' \backslash \{r, s\}$ is the square of four independent elements of G.*

Proof. Let $t \in G' \backslash \{r, s\}$, and suppose that t is the square of four independent elements z_1, z_2, z_3, z_4 of G. Since the squaring map on G is constant on cosets of G', we may assume that each z_i is a product of some elements of the basis $\mathcal{B}$. For $i = 1, \dots, 4$, let X_i and Y_i respectively denote the sets of elements of $\{x_1, \dots, x_k\}$ and $\{y_{k+1}, \dots, y_n\}$ that occur in z_i. We note that $|X_i \cup Y_i| \geq 2$ in each case, since $z_i^2 \notin \{r, s\}$. Comparing the four expressions for the common square of the elements z_i, we have

$$r^{|X_1|} s^{|Y_1|} C(X_1 \cup Y_1) = r^{|X_2|} s^{|Y_2|} C(X_2 \cup Y_2) = r^{|X_3|} s^{|Y_3|} C(X_3 \cup Y_3) = r^{|X_4|} s^{|Y_4|} C(X_4 \cup Y_4).$$

In each case, the expression $r^{|X_i|} s^{|Y_i|}$ is either equal to id, r, s or rs. No two of these can coincide, since the four elements $C(X_i \cup Y_i)$ of G' are distinct. After relabeling if necessary, we write

$$C(X_1 \cup Y_1) = rC(X_2 \cup Y_2) = sC(X_3 \cup Y_3) = rsC(X_4 \cup Y_4). \quad (3)$$

where $|X_1|$, $|Y_1|$, $|Y_2|$ and $|X_3|$ are even, and $|X_2|$, $|Y_3|$, $|X_4|$ and $|Y_4|$ are odd. Multiplying the expressions in (3) together, we obtain

$$C(X_1 \cup Y_1) C(X_4 \cup Y_4) = C(X_2 \cup Y_2) C(X_3 \cup Y_3). \quad (4)$$

Let V be a set of vertices corresponding to the elements of $\mathcal{B}$, and for $i = 1, \dots, 4$, let V_i be the subset of V corresponding to $X_i \cup Y_i$. Let Γ_i be the graph on vertex set V, whose edges form a complete graph on

V_i. The sets $V_1, \ldots, V_4$ are distinct, and each has at least two elements since $t \notin \{r, s\}$. The statement (4) translates to the following equality involving edge sets.

$$E(\Gamma_1) \triangle E(\Gamma_4) = E(\Gamma_2) \triangle E(\Gamma_3).$$

We write Γ for the graph consisting of the edges in $E(\Gamma_1) \triangle E(\Gamma_4)$, and their incident vertices. We note that Γ has at least three vertices, and that Γ is not a complete graph.

Let u and v be a pair of non-adjacent vertices in Γ. Then either u and v both belong to $V_2 \cap V_3$, or one of these vertices belongs to $V_2 \backslash V_3$ and the other to $V_3 \backslash V_2$. Moreover, either u and v both belong to $V_1 \cap V_4$, or one belongs to $V_1 \backslash V_4$ and the other to $V_4 \backslash V_1$.

Suppose that $\{u, v\} \subseteq V_2 \cap V_3$. Then u and v have the same set of neighbors in Γ, and this set is $V_2 \triangle V_3$. The subgraph of Γ induced on $V_2 \triangle V_3$ is complete (if $V_2 \supseteq V_3$ or $V_3 \supseteq V_2$), or consists of two complete components, on the disjoint sets $V_2 \backslash V_3$ and $V_3 \backslash V_2$. The set consisting of u, v and their non-neighbors in Γ is $V_2 \cap V_3$. Thus the sets V_2 and V_3 are determined by the non-adjacent pair $\{u, v\}$ and the hypothesis that $\{u, v\} \subseteq V_2 \cap V_3$. If, in addition, $\{u, v\} \subseteq V_1 \cap V_4$, then the same reasoning leads to the contradiction that $\{V_1, V_4\} = \{V_2, V_3\}$. Thus if $\{u, v\} \subseteq V_2 \cap V_3$, then we may assume that $u \in V_1 \backslash V_4$ and $v \in V_4 \backslash V_1$.

Similar reasoning leads from the hypothesis $u \in V_2 \backslash V_3$ and $v \in V_3 \backslash V_2$ to the conclusion that $\{u, v\} \subseteq V_1 \cap V_4$. In this case V_2 consists of u and its neighbors in Γ, and if $u \notin V_1 \cap V_4$ then either $V_1 = V_2$ or $V_4 = V_2$. Again we find in this situation that $\{V_2, V_3\} = \{V_1, V_4\}$.

We proceed with $u \in V_1 \backslash V_4$, $v \in V_4 \backslash V_1$, and $\{u, v\} \subseteq V_2 \cap V_3$. The vertices u and v have the same set of neighbors in Γ, which is $V_2 \triangle V_3$. It follows that $V_1 \backslash V_4 = \{u\}$ (since any other vertex in $V_1 \backslash V_4$ would be adjacent to u but not v in Γ) and that $V_4 \backslash V_1 = \{v\}$.

If any vertex of Γ belongs to all four of the Γ_i, then its neighbor set is simultaneously equal to $V_1 \triangle V_4$ and $V_2 \triangle V_3$. Since these two sets are different, it follows that $V_1 \cap V_2 \cap V_3 \cap V_4$ is empty, and $V_2 \cap V_3 \subseteq V_1 \triangle V_4 = \{u, v\}$. Hence $V_2 \cap V_3 = \{u, v\}$. Moreover, $V_1 \cap V_4 = V_2 \triangle V_3$. We may assume that $V_2 \backslash V_3$ includes an element x, since $V_2 \triangle V_3$ is not empty. Then $x \in V_1 \cap V_4$, and u and v are the only neighbors of x in Γ. It follows that $V_2 \backslash V_3 = \{x\}$. Similarly $V_3 \backslash V_2$ has at most one element. We have two possibilities.

1. $V_1 = \{u, x\}$, $V_4 = \{v, x\}$, $V_2 = \{u, v, x\}$, $V_3 = \{u, v\}$. In this case Γ is a path on three vertices, with edges ux and vx.
2. There is a single vertex y in $V_3 \backslash V_2$. In this case $V_1 = \{u, x, y\}$, $V_4 = \{v, x, y\}$, $V_2 = \{u, v, x\}$, $V_3 = \{u, v, y\}$. The graph Γ is a cycle of length 4, and it has two different representations as the symmetric difference of two copies of K_3.

Neither of these solutions satisfies the parity restrictions in (4), and we conclude that no element of $G' \backslash \{r, s\}$ can occur as the square of elements from more than three independent cosets of G' in G. □

We highlight the following immediate consequence of Theorem 3.5, which has a key role in our analysis.

Corollary 3.6. *The squares of the elements of a 2-square basis are uniquely determined in a covering group of C_2^n whose uniform corank is at least 4.*

We return now to the task of showing that every 2-uniform covering group possesses a 2-uniform basis. Suppose that G is a 2-uniform covering group of C_2^n, whose uniform rank is at least 4. Let $\{x_1, \ldots, x_m, y_{m+1}, \ldots, y_n\}$ be a basis of G, where $x_i^2 = r$ for $i \in \{1, \ldots, m\}$, and $y_j^2 = s \neq r$, for $j \in \{m+1, \ldots, n\}$. If $\rho(G) \in \{m, n-m\}$, then this is a 2-uniform basis of G. If not, let S be a set of $\rho(G)$ independent elements of G all having the same square. By Corollary 3.6, this common square must either be r or s, and after relabeling the elements of the generating set if necessary, we may assume that it is r. Then we may extend the set $\{x_1,\dots,x_m\}$ to a set $\{x_1,\dots,x_{\rho(G)}\}$ of independent elements of G with square r, discarding an element y_i from the original basis for each of the newly introduced elements $x_{m+1},\dots,x_{\rho(G)}$. The result is a 2-uniform basis of G.

It remains to consider the case where G is a 2-uniform covering group of C_2^n with $\rho(G)\leq 3$. In this case $n\leq 6$. Both covering groups of C_2^2 are uniform, so the cases of interest occur when $n\in\{3,4,5,6\}$. We first observe that if $\rho(G)=n-1$, then any set of $n-1$ independent elements with the same square can be extended to a 2-uniform basis by adding one further element. If $\rho(G)=\lceil\frac{n}{2}\rceil$, then every 2-square basis of G must have $\rho(G)$ elements with one square and $n-\rho(G)$ elements with the other. Every 2-square basis is therefore a 2-uniform basis. This observation accounts for the remaining cases, which occur when $(\rho(G),n)\in\{(2,4),(3,5),(3,6)\}$. We have proved the following statement.

Theorem 3.7. *If n is a positive integer and G is a 2-uniform covering group of C_2^n, then G possesses a 2-uniform basis.*

Theorem 3.7 allows us to restrict our attention to 2-colored graphs that arise from 2-uniform bases. We will refer to such graphs as *2-uniform graphs*, and give a descriptive characterization of them in terms of their graph-theoretic properties. We also establish conditions for the existence of a unique 2-uniform basis in a covering group. This step identifies a large class of covering groups that are represented by a unique 2-uniform graph. The exceptions to this situation will be categorized in this section, and analyzed later.

If G is a 2-uniform covering group of C_2^n, then the graph that represents G with respect to a 2-uniform basis has $\rho(G)$ vertices of one color, and $n-\rho(G)$ of the other. We adopt the convention that the color blue is used for $\rho(G)$ vertices representing basis elements with the same square, and red for the remainder. From now on, we will only consider graphs that are written with respect to 2-uniform bases, and thus only graphs that have at least as many blue as red vertices.

Definition 3.8. A 2-uniform graph is a 2-colored graph that represents a 2-uniform covering group with respect to a 2-uniform basis.

The remainder of this section discusses how to recognize a 2-uniform graph. We consider the question of how a 2-colored graph of order $n\geq 5$, with at least $\frac{n}{2}$ blue vertices, could fail to be 2-uniform. Suppose that $\mathcal{B}=\{x_1,\dots,x_k,y_{k+1},\dots,y_n\}$ is a 2-square basis of a covering group G of C_2^n, where $n\geq 5$, $k\geq\frac{n}{2}$, $x_i^2=r$ for each x_i, $y_j^2=s$ for each y_j, and $s\neq r$. We write $\mathcal{X}$ for $\{x_1,\dots,x_k\}$ and $\mathcal{Y}$ for $\{y_{k+1},\dots,y_n\}$. If $\rho(G)=3$, then $n\in\{5,6\}$ and $\mathcal{B}$ is a 2-uniform basis of G. If $\mathcal{B}$ is not a 2-uniform basis of G, then $\rho(G)\geq 4$ and $k<\rho(G)$. It follows from Theorem 3.5 that a 2-uniform basis of G possesses $\rho(G)$ elements with square r, or $\rho(G)$ elements with square s. This means either that $g^2=r$ for some $g\in G\backslash\langle x_1,\dots x_k\rangle$, or that $h^2=s$ for some $h\in G\backslash\langle y_{k+1},\dots,y_n\rangle$, and in the latter case that G contains enough independent elements h of this type to extend $\{y_{k+1},\dots,y_n\}$ to a set of $\rho(G)$ elements. Our next lemma establishes the circumstances under which such adjustments are possible.

Lemma 3.9. *Let G be a 2-uniform covering group of C_2^n, with a 2-square basis $\mathcal{B}$ as above. If none of the following conditions holds, then the maximum number of independent elements of G having square r is k. If exactly one of them holds, this number is $k+1$. If (b) and (c) hold with $S_b\cap\mathcal{Y}=S_c\cap\mathcal{Y}$, it is $k+1$. In other cases where two of the three conditions hold, it is $k+2$.*

(a) There is a subset S_a of $\mathcal{B}$, consisting of an even number of elements of $\mathcal{X}$ and a positive even number of elements of $\mathcal{Y}$, for which $r=C(S_a)$.

(b) There is a subset S_b of $\mathcal{B}$, consisting of an odd number of elements of $\mathcal{X}$ and an odd number of elements of $\mathcal{Y}$, for which $s=C(S_b)$.

(c) There is a subset S_c of $\mathcal{B}$, consisting of a positive even number of elements of $\mathcal{X}$ and an odd number of elements of $\mathcal{Y}$, for which $rs=C(S_c)$.

Proof. The maximum number of independent elements of G that have square r is the dimension of the vector subspace of G/G' spanned by all cosets consisting of elements with square r. Since the set of cosets represented by elements of $\mathcal{X}$ extends to a basis of this space, it is sufficient to consider whether G can include elements with square r that do not belong to the subgroup generated by $\mathcal{X}$ and G'.

If such an element x exists, we may assume that $x = s_1 s_1 \dots s_m$, where the s_i are elements of $\mathcal{B}$. We write $S = \{s_1, \dots, s_m\}$. Then

$$r = x^2 = r^e s^f C(S),$$

where $e = |S \cap \mathcal{X}|, f = |S \cap \mathcal{Y}|$, and $f \geq 1$ since $x \notin \langle \mathcal{X}, G' \rangle$. This equation is satisfied if and only if one of the following occurs:
(a) e and f are both even and $r = C(S)$;
(b) e and f are both odd and $r = rsC(S)$, so $s = C(S)$;
(c) e is even, f is odd and $r = sC(S)$, so $rs = C(S)$.
Each of these conditions can hold for at most one subset S of $\mathcal{B}$. Since there is no relation between r and s that is intrinsic to the definition of a 2-square basis, any pair of the three conditions may hold simultaneously, for a different subset S in each case. However, it is not possible for all three conditions to be satisfied. Suppose that the first two both hold, for respective subsets S_a and S_b of $\mathcal{B}$, each having at least two elements. If $|S_a \cap S_b| \geq 2$, let x and y be elements of $S_a \cap S_b$ and let $z \in S_a \triangle S_b$. Then $[x, z]$ and $[y, z]$ occur in rs, but $[x, y]$ does not, so rs cannot be represented by a clique as in (c). If $S_a \cap S_b = \{x\}$, then $S_a \backslash S_b$ and $S_b \backslash S_a$ are non-empty, with respective elements y and z. Then $[x, y]$ and $[x, z]$ occur in rs but $[y, z]$ does not, which is again inconsistent with (c). Finally if $S_a \cap S_b = \emptyset$, let $x, y \in S_a$ and $z, w \in S_b$. Then $[x, y]$ and $[z, w]$ occur in rs but $[x, z]$ does not, so rs does not have the form described in (c).

Each one of the three conditions (a), (b), (c) that holds in G yields an element of square r that is independent of $\{x_1, \dots, x_k\}$, represented by a product of the basis elements in the relevant set S. If both (b) and (c) hold with $S_b \cap \mathcal{Y} = S_c \cap \mathcal{Y}$ then the process yields only $k+1$ independent elements that can occur together in a basis. Otherwise, if two of the three conditions hold, we obtain $k+2$ independent elements with square r. □

Applying Lemma 3.9 to the element s instead of r, we note that the number of independent elements of G whose square is s is at most $n-k+2$, and the value of this number is determined by the conditions (a), (b), (c) in the statement of the lemma, with the roles of $\mathcal{X}$ and $\mathcal{Y}$ reversed. The conditions of Lemma 3.9 may be expressed as properties of the graph $\Gamma_{\mathcal{B}}(G)$ and used to characterize 2-uniform graphs. Before proceeding with this description, we introduce some notation that will apply to 2-colored graphs in general.

For a 2-colored graph Γ, we write Γ^B and Γ^R, respectively for the subgraphs of Γ whose edge sets are the sets of blue and red edges, on their respective sets of incident vertices. We write $\Gamma^{B \triangle R}$ for the subgraph of Γ whose edge set is $E(\Gamma^B) \triangle E(\Gamma^R)$, on the vertices incident with these edges, with each edge retaining its color in Γ. We write $\Gamma^\star$ for the *color opposite* of Γ, which is obtained from Γ by switching the color of every vertex and every edge, from blue to red or from red to blue. It is clear that Γ and $\Gamma^\star$ represent the same group G, with respect to the same two-square basis.

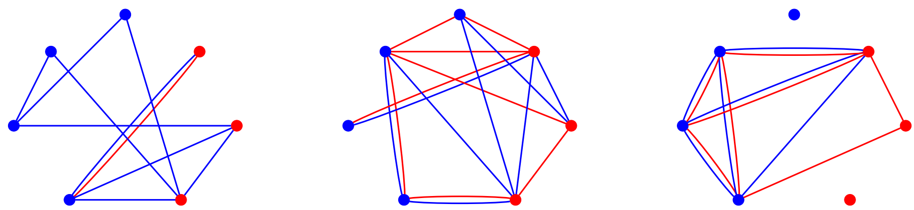
The following description of 2-uniform graphs now follows from Lemma 3.9.

Theorem 3.10. *Let Γ be a 2-colored graph on at least 5 vertices, with at least as many blue vertices as red. Then Γ is a 2-uniform graph if and only if the following conditions hold.*

(a) Γ^B is not a clique on an even number of blue vertices and a positive even number of red vertices;
(b) Γ^R is not a clique on an odd number of blue vertices and an odd number of red vertices;
(c) $\Gamma^{B \triangle R}$ is not a clique on an even number of blue vertices and an odd number of red vertices;
(d) If the numbers of blue and red vertices in Γ are equal, then items (a), (b), and (c) above apply to the color opposite $\Gamma^\star$ of Γ.

(e) *If the numbers of blue and red vertices in Γ differ by 1, then $\Gamma^\star$ fails at most one of conditions (a), (b), (c), or fails both (b) and (c) with cliques involving the same set of red vertices.*

Example 3.11. These three 2-colored graphs, each having more blue than red vertices, all fail to be 2-uniform graphs, respectively on the basis of items (b), (c), and (e) of Theorem 3.10.



4. Exchange operations on 2-uniform graphs

Our ambition is to construct a bijective correspondence between isomorphism classes of 2-uniform covering groups of C_2^n, and an appropriate collection of 2-colored graphs of order n. A graph is constructed not intrinsically from a group, but from a 2-square basis. As Example 3.2 indicates, a covering group of C_2^n may have multiple 2-square bases, possibly even corresponding to graphs whose vertex colorings partition n differently.

Theorem 3.10 gives a full description of 2-uniform graphs of order 5 or greater. We now consider the question of when non-isomorphic 2-uniform graphs describe isomorphic groups. This requires that the graphs have the same numbers of blue and red vertices, since the number of blue vertices is the uniform rank, an invariant of the group. The remainder of the article is devoted to the question of when a 2-uniform covering group of an elementary abelian 2-group has multiple 2-uniform bases, determining non-isomorphic 2-uniform graphs. We remark that this always occurs in the case of a 2-uniform graph of C_2^n of uniform corank 1, since a set of $n-1$ independent elements can be extended to a 2-uniform basis by the addition of any element from outside their span. The special case of corank 1 will be discussed in Section 7; in the meantime we restrict attention to 2-uniform covering groups whose uniform corank is at least 2.

We begin by considering the possibility that a group has multiple distinct 2-uniform bases involving elements with the same pair of squares r and s.

Lemma 4.1. *Let G be a covering group, with 2-square basis $\mathcal{B} = \{x_1, \dots, x_k, y_{k+1}, \dots, y_n\}$, where $x_i^2 = r$, $y_j^2 = s \neq r$, and where k and $n-k$ are both at least 2. Suppose that $\mathcal{B}' = (\mathcal{B} \cup \{z\}) \setminus \{w\}$ is another 2-square basis of G, where $z \in G$ with $z^2 \in \{r, s\}$, no element of zG' belongs to $\mathcal{B}$, and $w \in \mathcal{B}$. Then at least one of the following occurs.*

1. *$\Gamma_{\mathcal{B}}^B$ is a clique on an even number of blue and an even number of red vertices.*
2. *$\Gamma_{\mathcal{B}}^R$ is a clique on an even number of blue and an even number of red vertices.*
3. *$\Gamma_{\mathcal{B}}^B$ is a clique on an odd number of blue and an odd number of red vertices.*
4. *$\Gamma_{\mathcal{B}}^R$ is a clique on an odd number of blue and an odd number of red vertices.*
5. *$\Gamma_{\mathcal{B}}^{B\triangle R}$ is a clique on an even number of blue and odd number of red vertices.*
6. *$\Gamma_{\mathcal{B}}^{B\triangle R}$ is a clique on an odd number of blue and an even number of red vertices.*

Proof. We may assume that z is a product of at least two elements of $\mathcal{B}$, in specified order. Relabeling as necessary, we write $z = x_1 \dots x_p y_{k+1} \dots y_{k+q}$. We write S for the subset $\{x_1, \dots, x_p, y_{k+1}, \dots, y_{k+q}\}$ of $\mathcal{B}$. Then

$$z^2 = r^p s^q C(S).$$

Suppose first that $z^2 = r$. We consider the parities of p and q. Since $|S| \geq 2$, it is not possible for p to be odd and q even. This leaves the remaining possibilities and outcomes.

- If p and q are both even, then $z^2 = C(S) = r$, corresponding to Item 1.
- If p and q are both odd, then $z^2 = rsC(S) = r$, so $s = C(S)$, corresponding to Item 4.
- If p is even and q odd, then $z^2 = sC(S) = r$, and $C(S) = rs$, corresponding to Item 5.

In the alternative case where $z^2 = s$, the possibility that p is even and q odd is excluded, and we have the following possibilities.

- If p and q are both even, then $z^2 = C(S) = s$, corresponding to Item 2.
- If p and q are both odd, then $z^2 = rsC(S) = s$, so $r = C(S)$, corresponding to Item 3.
- If p is odd and q even, then $z^2 = rC(S) = s$, and $C(S) = rs$, corresponding to Item 6. □

In each of the six cases of Lemma 4.1, any element w of S can be eliminated from $\mathcal{B} \cup \{z\}$ to form the alternative 2-square basis $\mathcal{B}'$. If $\mathcal{B}' = (\mathcal{B} \cup \{z\}) \setminus \{w\}$, a description of the relationship between the edge sets of the graphs $\Gamma_{\mathcal{B}'}$ and $\Gamma_{\mathcal{B}}$ is provided by a direct application of Theorem 2.7. The vertex sets may differ by the color of a single vertex, if the elements w and z have different squares. These general considerations may be applied to all 2-square bases. Our interest however is in the case of 2-uniform graphs, in which the number of blue vertices coincides with the uniform rank of the associated group, and is thus maximal among all 2-colored graphs representing that group. If $\Gamma_{\mathcal{B}}$ is a 2-uniform graph, a basis change of the type described above cannot replace a red vertex with a blue one; graphs that admit this possibility are excluded by Theorem 3.10. Basis changes that replace a blue vertex with a red one do not preserve the 2-uniform property and are thus not of interest (except in the case where the numbers of blue and red vertices differ by 1, which is considered below).

For a 2-uniform graph of uniform corank at least 2, we refer to the operation of adjusting one 2-uniform basis to another, by replacing a single element, as an *exchange* operation. We refer to the transition between their corresponding graphs as an exchange operation of graphs, where we assume that both graphs have the same vertex set, with a single vertex relabeled in the transition. In Theorem 4.2, we give a graph-theoretic description of the exchange operations on 2-uniform graphs that preserve the color of the relabeled vertex (and hence preserve the 2-uniform property). We refer to exchanges of this type as *simple exchanges*.

Theorem 4.2 is the result a straightforward application of Theorem 2.7 and Corollary 2.8 to the simple exchange possibilities that preserve the 2-uniform property for graphs, as outlined in Theorem 3.10. Before stating it, we introduce some notation for describing the neighbor set of a vertex, via colored or uncolored edges.

For a vertex v of a 2-colored graph Γ, we write $E(v)$ for the set $E(N^B(v), N^R(v))$, where $N^B(v)$ and $N^R(v)$ respectively denote the sets of neighbors of v in Γ, via blue and red edges. If $N^{B\setminus R}(v)$ denotes the set of vertices of Γ that are adjacent to v via blue edges only, and $N^{R\setminus B}(v)$ and $N^{R\cap B}(v)$ are similarly defined, then $E(v)$ is the edge set of the complete tripartite (or bipartite or null) graph with parts $N^{B\setminus R}(v)$, $N^{R\setminus B}(v)$ and $N^{B\cap R}(v)$. We consider $E(v)$ itself to be a set of uncolored edges, and write $E^R(v)$ and $E^B(v)$ respectively for the same set of edges, all colored red or all blue.

Theorem 4.2. *Let Γ be a 2-uniform graph of order n, with at least two red vertices, describing a 2-uniform covering group G of C_2^n, with respect to a basis $\mathcal{B}$. An alternative 2-uniform graph Γ', describing G with respect to a basis obtained from G by a simple exchange operation, may arise under the following conditions and in the following ways. In all cases we consider that Γ and Γ' have the same vertex set, with a single vertex relabeled in the transition from one graph to the other.*

1. *(Type 1) If Γ^B is a clique on an even number of blue vertices, then $E(\Gamma')$ may be given by $E(\Gamma) \triangle E^R(v)$ for any vertex v of the clique.*
2. *(Type 2) If Γ^R is a clique on an even number of blue vertices and a positive even number of red vertices, then $E(\Gamma')$ may be given by $E(\Gamma) \triangle E^B(v)$, for any red vertex v of the clique.*

3. *(Type 3) If Γ^B is a clique on an odd number of blue and an odd number of red vertices, then $E(\Gamma')$ may be given by $E(\Gamma)\triangle E^R(v)$, for any red vertex v of the clique.*
4. *(Type 4) If $\Gamma^{B\triangle R}$ is a clique on an odd number of blue and an even number of red vertices, then $E(\Gamma')$ may be given by $E(\Gamma)\triangle E^R(v)\triangle E^B(v)$, for any red vertex v of the clique.*

It remains to consider exchange operations that switch the vertex colors. If the uniform rank k of G exceeds the uniform corank $n-k$ by only 1 or 2, then Lemma 3.9 gives conditions under which G may possess a 2-uniform basis having k elements of square s and $n-k$ of square r. We conclude this section by giving a description of the corresponding graph operations in such cases.

First suppose that $k = (n-k)+1$. Suppose that exactly one of the conditions of Lemma 3.9 holds. Then G contains an element z of square s that is independent of $\{y_{k+1}, \dots, y_n\}$. In the graph $\Gamma_B(G)$, either the blue edge set, or the red edge set, or their symmetric difference, forms a clique on those vertices representing the elements of $\mathcal{B}$ that occur in z. We may adjust $\mathcal{B}$ to another 2-uniform basis $\mathcal{B}'$ by replacing an element x_i of square r with z. Then $\mathcal{B}'$ has k elements of the square s and $k-1$ of square r. In a 2-uniform graph Γ' corresponding to $\mathcal{B}'$, elements of square s and r are respectively represented by blue and red vertices (opposite to the situation in Γ). The following theorem describes transformations of 2-uniform graphs corresponding to exchanges of this type.

Theorem 4.3. *Let Γ be a 2-uniform graph of order $2k-1$, with k blue vertices and $k-1$ red vertices. Alternative 2-uniform graphs Γ' describing the same group may arise in the following ways.*

1. *If Γ^R is a clique on a positive even number of blue vertices and an even number of red vertices, we may choose a blue vertex v of this clique, transform Γ to Γ_1 by switching the color of v from blue to red, and then define Γ' to be the color opposite of the graph with edge set $E(\Gamma_1)\triangle E^B(v)$.*
2. *If Γ^B is a clique on a odd number of blue vertices and an odd number of red vertices, we may choose a blue vertex v of this clique, transform Γ to Γ_1 by switching the color of v from blue to red, and then define Γ' to be the color opposite of the graph with edge set $E(\Gamma_1)\triangle E^R(v)$.*
3. *If $\Gamma^{B\triangle R}$ is a clique on an odd number of blue vertices and an even number of red vertices, we may choose a blue vertex v of this clique, transform Γ to Γ_1 by switching the color of v from blue to red, and then define Γ' to be the color opposite of the graph with edge set $E(\Gamma_1)\triangle E^R(v)\triangle E^B(v)$.*

Theorem 4.3 is proved by direct application of Theorem 2.7 and Corollary 2.8.

Finally, if $k=(n-k)+2$, and exactly two of the three conditions of Theorem 4.3 hold in Γ (involving different sets of blue vertices), we can increase of independent elements of square s by 2, to obtain a 2-uniform basis in which the number of elements of square s is the uniform rank k. We refer to a change of basis of this nature as a *double exchange*. Let the 2-uniform graph Γ, with vertex set V, corresponding to a 2-uniform basis of a covering group G, with k elements of square r represented by the blue vertices, and $k-2$ vertices of square s represented by the red vertices. Let Γ_1 and Γ_2, with vertex sets V_1 and V_2, respectively, be the subgraphs of Γ that respectively satisfy two of the three conditions in Theorem 4.3, and let c_1 and c_2 be the elements of G' represented by the edge sets of the cliques Γ_1 and Γ_2. Then $\{c_1, c_2\} \subset \{r, s, rs\}$.

A double exchange operation from Γ to Γ' begins with the selection of a blue vertex v_1 of the clique Γ_1, and a blue vertex v_2 of the clique Γ_2, representing elements x_1 and x_2 of a basis $\mathcal{B}$. In the alternative basis $\mathcal{B}'$, x_1 and x_2 are respectively replaced by z_1 and z_2, which are the products of the elements of $\mathcal{B}$ represented respectively by the vertices of Γ_1 and Γ_2. A necessary condition for $\mathcal{B}'$ to generate the group is that the vertices v_1 and v_2 do not both belong to both Γ_1 and Γ_2. We may assume that Γ_1 includes the vertex v_1 and not v_2.

Since v_2 is incident with no edge of Γ_1, it follows from Corollary 2.8 that the set of edges representing c_1 is the same for both bases. We apply Theorem 2.9 to c_2. The sets P_2 and Q_2 coincide; both are equal to $V_2\backslash\{v_2\}$. If v_1 is incident with no edge of Γ_2, then P_2 is empty and c_2 is described by the same set of edges with respect to both bases, by item 1. of Theorem 2.9.

If the vertex v_1 belongs to the clique Γ_2, then Item 4 of Theorem 2.9 applies, and (since $P_2 = Q_2$), it asserts that the edge sets that represent c_2 with respect to the two bases differ by $E(P_1, Q_1) = E(v_1)$, where P_1 and Q_1 are respectively the sets of neighbors of v_1 in Γ_1 and Γ_2. The color(s) of the adjusted edges depends on whether c_2 coincides with the element r, s or rs.

The following statement summarizes the double exchange operation on graphs.

Theorem 4.4. *Let Γ be a 2-uniform graph in which the numbers of blue and red vertices differ by 2. Suppose that Γ satisfies exactly two of the three conditions of Theorem 4.3, on cliques Γ_1 and Γ_2, with vertex sets V_1 and V_2 respectively, involving different sets of red vertices. Let v_1 and v_2 be blue vertices of Γ_1 and Γ_2 respectively, where v_2 does not belong to Γ_1. Let Φ be the graph obtained from Γ by recoloring the vertices v_1 and v_2 from blue to red, and adjusting the edge set as follows:*

1. *If v_1 does not belong to Γ_2, then $E(\Phi) = E(\Gamma)$.*
2. *If v_1 belongs to Γ_2 and $\Gamma_2 = \Gamma^R$, then $E(\Phi) = E(\Gamma)\triangle E^R(v_1)$.*
3. *If v_1 belongs to Γ_2 and $\Gamma_2 = \Gamma^B$, then $E(\Phi) = E(\Gamma)\triangle E^B(v_1)$.*
4. *If v_1 belongs to Γ_2 and $\Gamma_2 = \Gamma^R$, then $E(\Phi) = E(\Gamma)\triangle E^R(v_1)\triangle E^B(v_1)$.*

Then the color opposite of Φ is a 2-uniform graph representing the same covering group as Γ.

5. Groups of uniform corank 3

Section 4 gives an account of those 2-uniform covering groups of C_2^n that admit multiple 2-uniform bases consisting of elements with the same pair of squares. By Corollary 3.6, if r and s are the squares of the elements of a 2-uniform basis of a covering group G of corank at least 4, then *every* 2-uniform basis of G consists of elements with squares r and s, and may be obtained from $\mathcal{B}$ through a sequence of exchange operations of the types described in Section 4. In the case of a 2-uniform covering group of C_2^n whose uniform rank k is at least $n-3$, a 2-uniform basis $\mathcal{B}$ consists of k elements with square r and up to three elements with a different square s. If $n-k \leq 3$, Theorem 3.5 leaves open the possibility that some element s' of G', with $s' \notin \{r, s\}$ could be the square of $n-k$ independent elements of G. If this occurs, an alternative 2-uniform basis of G might be obtained from $\mathcal{B}$ by replacing the elements of square s with elements of square s'. This certainly occurs in the case $n-k=1$, where a set of $n-1$ independent elements with the same square may be extended to a 2-uniform basis by the addition of any element outside their span.

In this section we consider the possibility of multiple choices for the element s, in the case of groups of uniform corank 3. Our analysis is presented subject to the assumption that $n \geq 7$, but can easily be extended to the case of groups whose uniform rank and corank are both equal to 3. In this case all considerations apply to the color opposite of all graphs in question, as well to the graphs themselves.

Let G be a 2-uniform covering group of C_2^n of uniform corank 3, where $n \geq 7$. Let $\mathcal{B} = \{x_1, \ldots, x_k, y_1, y_2, y_3\}$ be a 2-uniform basis of G, where $x_i^2 = r$ and $y_i^2 = s$, $r \neq s$. We write $\mathcal{X}$ and $\mathcal{Y}$ for the subsets $\{x_1, \ldots, x_k\}$ and $\{y_1, y_2, y_3\}$ of $\mathcal{B}$. By Theorem 3.5, no element of G', except r and possibly s, is the square of more than three independent elements of G'. We now establish the conditions under which $\mathcal{B}$ may be adjusted to a new 2-uniform basis $\mathcal{B}'$, by replacing y_1, y_2, y_3 with independent elements z_1, z_2, z_3 having the same square s', where $s' \neq s$.

Suppose that z_1, z_2, z_3 are elements of G with these properties. Since the squaring map in G is constant on cosets of G', we may assume that each of z_1, z_2, z_3 is the product of some elements of $\mathcal{B}$. We write $\mathcal{Z}_i$ for the set of elements of $\mathcal{B}$ that occur in z_i. For each i, we may write

$$s' = z_i^2 = r^{|\mathcal{Z}_i \cap \mathcal{X}|} s^{|\mathcal{Z}_i \cap \mathcal{Y}|} C_i, \tag{5}$$

where $C_i = C(\mathcal{Z}_i)$. Since C_1, C_2 and C_3 are distinct elements of G', their prefixes $r^{|\mathcal{Z}_i \cap \mathcal{X}|}s^{|\mathcal{Z}_i \cap \mathcal{Y}|}$ must also be distinct for $i = 1, 2, 3$.

After relabeling if necessary, we may assume that $|\mathcal{Z}_1 \cap \mathcal{Y}|$ and $|\mathcal{Z}_2 \cap \mathcal{Y}|$ have the same parity. Then $|\mathcal{Z}_1 \cap \mathcal{X}|$ and $|\mathcal{Z}_2 \cap \mathcal{X}|$ have opposite parity. Comparing the descriptions of z_1^2 and z_2^2 in (5), we find that $r = C_1C_2$, where C_1 and C_2 are elements of G' whose graphs with respect to $\mathcal{B}$ are nontrivial cliques, whose numbers of blue vertices have opposite parity, and whose numbers of red vertices have the same parity. Now $|\mathcal{Z}_3 \cap \mathcal{X}|$ has the same parity as exactly one of $|\mathcal{Z}_1 \cap \mathcal{X}|$ and $|\mathcal{Z}_2 \cap \mathcal{X}|$; we may assume this to be $|\mathcal{Z}_2 \cap \mathcal{X}|$, after relabeling again if necessary. We note that $|\mathcal{Z}_3 \cap \mathcal{Y}|$ and $|\mathcal{Z}_2 \cap \mathcal{Y}|$ have opposite parity. Comparing the expressions for z_2^2 and z_3^2 in (5) gives $s = C_2C_3$, where $C_3 \in G'$ is represented on the vertex set of $\Gamma_{\mathcal{B}}$ by a clique whose numbers of blue and red vertices are respectively of the same and opposite parity to those of the graph representing C_2.

The following lemma notes the meaning of these observations in terms of a 2-uniform graph representing G. We note that a graph satisfying the conditions of Lemma 5.1 cannot also satisfy the conditions in any of Theorems 4.2, 4.3, or 4.4. If a 2-uniform covering group of corank 3 has multiple 2-uniform bases related by exchange operations of the types described in Section 4, the same group cannot have multiple 2-uniform bases related by the considerations in this section.

Lemma 5.1. *Let Γ be a 2-uniform graph of order $n \geq 7$, with three red vertices. Let G be the 2-uniform covering group of C_2^n with basis $\mathcal{B}$ determined by Γ. Then G contains elements z_1, z_2, z_3 representing different cosets of G' and all having the same square s', with $s' \notin \{r, s\}$ if and only if the following conditions hold in Γ.*

1. *$E(\Gamma^B) = E(\Phi_1) \triangle E(\Phi_2)$, where Φ_1 and Φ_2 are nontrivial cliques whose numbers of blue vertices have opposite parity and whose numbers of red vertices have the same parity, and;*
2. *$E(\Gamma^R) = E(\Phi_2) \triangle E(\Phi_3)$, where Φ_3 is a nontrivial clique whose numbers of blue and red vertices respectively have the same and opposite parity to the corresponding numbers in Φ_2.*

If these conditions are satisfied, let z_i be the product in G of the basis elements represented by the vertices of Φ_i (in any order). Then $z_1^2 = z_2^2 = z_3^2$.

For a graph Γ satisfying the conditions of Lemma 5.1, it is not automatic that the elements z_1, z_2, z_3 are independent of the $n - 3$ basis elements represented by blue vertices in Γ. This requires a linear independence condition which we express in matrix terms as follows. Let v_1, v_2, v_3 be labels on the red vertices of Γ. Define a 3×3 matrix $B \in M_3(\mathbb{F}_2)$ whose (i, j) entry is 1 if the vertex v_j occurs in the clique Φ_i, and 0 otherwise. Then $\{z_1, z_2, z_3\}$ extends the set of elements of $\mathcal{B}$ represented by blue vertices in Γ to a 2-uniform basis $\mathcal{B}'$ of G, if and only if B is nonsingular in $M_3(\mathbb{F}_2)$.

Our theme for the remainder of this section is a description of the relationship between the graphs determined by the 2-uniform bases $\mathcal{B}$ and $\mathcal{B}'$ of G, when the matrix B is nonsingular.

We begin with some remarks on the uniqueness of Φ_1, Φ_2, and Φ_3, under the conditions of Lemma 5.1. This involves the application of Theorem 3.5 and its proof. It was shown there that the edge set of any graph has at most one expression as the symmetric difference of the edge sets of two cliques, with the two exceptions of the path P_3 on 3 vertices, and the cycle C_4 on four vertices. Each of these has two expressions as the symmetric difference of a pair of cliques. Under the conditions of Lemma 5.1, the question of alternative possibilities for the Φ_i (and hence the z_i) arises only if Γ^B or Γ^R is a copy of P_3 or C_4. For both P_3 and C_4, it is routine to check that there is no coloring of the vertices that yields multiple decompositions satisfying both the parity conditions of Lemma 5.1 and the requirement that the 3×3 matrix B is nonsingular. We conclude that if $\mathcal{B} = \{x_1, \ldots, x_{n-3}, y_1, y_2, y_3\}$ is a 2-uniform basis of a covering group G of C_2^n of corank 3, with $x_i^2 = r$ and $y_i^2 = s \neq r$, then there is at most one choice for a set $\{z_1G', z_2G', z_3G'\}$, with the property that $\mathcal{B}' = \{x_1, \ldots, x_{n-3}, z_1, z_2, z_3\}$ is an alternative 2-uniform basis of G', where $z_i^2 = s' \neq s$.

We now assume that G is a covering group of corank 3 of C_2^n, possessing 2-uniform bases $\mathcal{B}$ and $\mathcal{B}'$ as above. We write P for the change of basis matrix from $\mathcal{B}'$ to $\mathcal{B}$, whose jth column records the $\mathcal{B}$-coordinates of the jth element of $\mathcal{B}'$. The first $n-3$ columns of P coincide with those of the identity matrix, and the last three columns, respectively correspond to z_1, z_2, z_3, which we assume to be ordered according to the description in Lemma 5.1. Thus P has the following form, where v_1, v_2, v_3 are vectors in $\mathbb{F}_2^{n-3}$, with the property that the numbers of entries equal to 1 in v_1 and v_2 have opposite parity, and the numbers of entries equal to 1 in v_2 and v_3 have the same parity. The lower right block B is a nonsingular matrix in $M_3(\mathbb{F}_2)$, with the property that the numbers of entries equal to 1 in its first two columns have the same parity, and the number of entries equal to 1 in its third column has the opposite parity to these.

$$P = \left[\begin{array}{c|ccc} I_{n-3} & \mid & \mid & \mid \\ & v_1 & v_2 & v_3 \\ & \mid & \mid & \mid \\ \hline 0_{(n-3)\times 3} & & B_{3\times 3} & \end{array}\right]. \tag{6}$$

The graph $\Gamma_{\mathcal{B}}(G)$ can be constructed from P as follows. For $i = 1, 2, 3$, we write E_i for the edge set of the clique on the set of vertices representing those elements of $\mathcal{B}$ where a 1 occurs in column $(n-3)+i$ of P; i.e. those elements of $\mathcal{B}$ that occur in z_i. The set of blue edges in $\Gamma_{\mathcal{B}}(G)$ is $E_1 \triangle E_2$, and the set of red edges is $E_2 \triangle E_3$. The change of basis matrix from $\mathcal{B}$ to $\mathcal{B}'$ is the inverse of P in $M_n(\mathbb{F}_2)$, given by

$$P^{-1} = \left[\begin{array}{c|ccc} I_{n-3} & \mid & \mid & \mid \\ & v_1B^{-1} & v_2B^{-1} & v_3B^{-1} \\ & \mid & \mid & \mid \\ \hline 0_{(n-3)\times 3} & & B^{-1}_{3\times 3} & \end{array}\right]. \tag{7}$$

The graph $\Gamma_{\mathcal{B}'}$ that represents G with respect to $\mathcal{B}'$ can be constructed from P^{-1} as $\Gamma_{\mathcal{B}}$ is from P. The subgraph $\Gamma_{\mathcal{B}'}^B$ comprising its blue edges has the form $\Psi_1 \triangle \Psi_2$, where Ψ_1 and Ψ_2 are cliques whose numbers of red vertices have the same parity and whose numbers of blue vertices have opposite parity. If we assume $\Gamma_{\mathcal{B}}$ and $\Gamma_{\mathcal{B}'}$ to have the same vertex set (with the red vertices labeled differently), the vertices of the cliques Ψ_1 and Ψ_2 are written in some pair of the last three columns of P^{-1}; these are the two columns in which the numbers of 1s among the last three entries have the same parity. Similarly, $\Gamma_{\mathcal{B}'}^R = \Psi_2 \triangle \Psi_3$, where the clique Ψ_3 is described by the remaining columns of P^{-1}, which also contains sufficient information to distinguish Ψ_1 from Ψ_2, on the basis that that the numbers of blue vertices in Ψ_2 and Ψ_3 have the same parity.

We now detail the transformations from $\Gamma_{\mathcal{B}}$ to $\Gamma_{\mathcal{B}'}$ corresponding to the distinct possibilities for the matrix B in the lower right 3×3 block of the matrix P. In the following analysis of these cases, we write S, T, U respectively for the vertex sets of the cliques Φ_1, Φ_2, and Φ_3, and use the superscripts B and R to denote their sets of blue and red vertices. We may reorder the elements y_1, y_2, y_3 in $\mathcal{B}$ as necessary, to ensure that the 3×3 matrix B in the lower right block of P has one of the following standard forms. Each of these forms occurs in two versions, depending on whether $|S^B|$, which is the number of 1s in v_1, is even or odd. We have a total of 16 cases, some pairs of which are equivalent under the transition between the two bases. In order to distinguish the cases on the basis of the graph $\Gamma_{\mathcal{B}}$, we generally require the expression for the sets of blue and red edges as symmetric differences of cliques.

1. $B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Case 1.(a): $|S^B|$ is odd. Case 1.(b): $|S^B|$ is even.

2. $B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Case 2.(a): $|S^B|$ is odd. Case 2.(b): $|S^B|$ is even.
3. $B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Case 3.(a): $|S^B|$ is odd. Case 3.(b): $|S^B|$ is even.
4. $B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. Case 4.(a): $|S^B|$ is odd. Case 4.(b): $|S^B|$ is even.
5. $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Case 5.(a): $|S^B|$ is odd. Case 5.(b): $|S^B|$ is even.
6. $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Case 6.(a): $|S^B|$ is odd. Case 6.(b): $|S^B|$ is even.
7. $B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Case 7.(a): $|S^B|$ is odd. Case 7.(b): $|S^B|$ is even.
8. $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Case 8.(a): $|S^B|$ is odd. Case 8.(b): $|S^B|$ is even.

We now analyze the transformation between $\Gamma_{\mathcal{B}}$ and $\Gamma_{\mathcal{B}'}$ in all cases.

1. In Case 1, we write P as in (6) and observe

$$P^{-1} = \left[\begin{array}{c|ccc} I_{n-3} & \begin{matrix}|\\ v_3 \\ |\end{matrix} & \begin{matrix}|\\ v_1+v_3 \\ |\end{matrix} & \begin{matrix}|\\ v_1+v_2+v_3 \\ |\end{matrix} \\ \hline & 0 & 1 & 1 \\ 0_{(n-3)\times 3} & 0 & 0 & 1 \\ & 1 & 1 & 1 \end{array}\right].$$

After reordering the last three columns and last three rows to obtain a standard form as above, we have the following descriptions of the change of basis matrix from $\mathcal{B}$ to $\mathcal{B}'$, respectively for Cases 1(a) and 1(b).

$$1.\text{(a)} \left[\begin{array}{c|ccc} I_{n-3} & \begin{matrix}|\\ v_3 \\ |\end{matrix} & \begin{matrix}|\\ v_1+v_2+v_3 \\ |\end{matrix} & \begin{matrix}|\\ v_1+v_3 \\ |\end{matrix} \\ \hline & 1 & 1 & 1 \\ 0_{(n-3)\times 3} & 0 & 1 & 0 \\ & 0 & 1 & 1 \end{array}\right]$$

$$1.\text{(b)} \left[\begin{array}{c|ccc} I_{n-3} & \begin{matrix}|\\ v_1+v_2+v_3 \\ |\end{matrix} & \begin{matrix}|\\ v_3 \\ |\end{matrix} & \begin{matrix}|\\ v_1+v_3 \\ |\end{matrix} \\ \hline & 1 & 1 & 1 \\ 0_{(n-3)\times 3} & 1 & 0 & 0 \\ & 1 & 0 & 1 \end{array}\right]$$

The matrices above are of types 7(b) and 8(b) respectively, and we conclude that Cases 1(a) and 1(b) are respectively equivalent to 7(b) and 8(b), in terms of the covering groups that they describe.

2. In Case 2,

$$P^{-1} = \left[\begin{array}{c|ccc} & \vert & \vert & \vert \\ I_{n-3} & v_1+v_3 & v_3 & v_2+v_3 \\ & \vert & \vert & \vert \\ \hline & 1 & 0 & 0 \\ 0_{(n-3)\times 3} & 0 & 0 & 1 \\ & 1 & 1 & 1 \end{array}\right].$$

After reordering the last three columns and last three rows to obtain a standard form as above, we have the following descriptions of the change of basis matrix from $\mathcal{B}$ to $\mathcal{B}'$, respectively for Cases 2(a) and 2(b).

$$2.(a) \left[\begin{array}{c|ccc} & \vert & \vert & \vert \\ I_{n-3} & v_1+v_3 & v_2+v_3 & v_3 \\ & \vert & \vert & \vert \\ \hline & 1 & 0 & 0 \\ 0_{(n-3)\times 3} & 1 & 1 & 1 \\ & 0 & 1 & 0 \end{array}\right] \quad 2.(b) \left[\begin{array}{c|ccc} & \vert & \vert & \vert \\ I_{n-3} & v_2+v_3 & v_1+v_3 & v_3 \\ & \vert & \vert & \vert \\ \hline & 1 & 0 & 0 \\ 0_{(n-3)\times 3} & 1 & 1 & 1 \\ & 0 & 1 & 0 \end{array}\right]$$

The matrices above are again of types 2(a) and 2(b) respectively, for these cases the graphs with respect to both $\mathcal{B}$ and $\mathcal{B}'$ are of the same type, 2(a) or 2(b). In these cases, the graphs $\Gamma_{\mathcal{B}}$ and $\Gamma_{\mathcal{B}'}$ are related in Case 2(a) by

$$V(\Psi_1) = V(\Phi_1)\triangle V^B(\Phi_3), V(\Psi_2) = V(\Phi_2)\triangle V^B(\Phi_3),\ V(\Psi_3) = V(\Phi_3),$$

and in Case 2(b) by

$$V(\Psi_1) = V(\Phi_2)\triangle V^B(\Phi_3), V(\Psi_2) = V(\Phi_1)\triangle V^B(\Phi_3),\ V(\Psi_3) = V(\Phi_3).$$

3. In Case 3,

$$P^{-1} = \left[\begin{array}{c|ccc} & \vert & \vert & \vert \\ I_{n-3} & v_1+v_2+v_3 & v_2+v_3 & v_3 \\ & \vert & \vert & \vert \\ \hline & 1 & 0 & 0 \\ 0_{(n-3)\times 3} & 1 & 1 & 0 \\ & 1 & 1 & 1 \end{array}\right].$$

In Cases 3(a) and 3(b), this may be adjusted to the following standard forms

$$3.(a) \left[\begin{array}{c|ccc} & \vert & \vert & \vert \\ I_{n-3} & v_1+v_2+v_3 & v_3 & v_2+v_3 \\ & \vert & \vert & \vert \\ \hline & 1 & 1 & 1 \\ 0_{(n-3)\times 3} & 1 & 0 & 0 \\ & 1 & 0 & 1 \end{array}\right]$$

$$3.(b) \left[\begin{array}{c|ccc} & \vert & \vert & \vert \\ I_{n-3} & v_3 & v_1+v_2+v_3 & v_2+v_3 \\ & \vert & \vert & \vert \\ \hline & 1 & 1 & 1 \\ 0_{(n-3)\times 3} & 0 & 1 & 0 \\ & 0 & 1 & 1 \end{array}\right]$$

The matrices above are of types 8(a) and 7(a), respectively, and we conclude that Cases 3(a) and 3(b) are respectively equivalent to 8(a) and 7(a), in terms of the covering groups that they describe.

4. In Case 4,

$$P^{-1} = \left[\begin{array}{c|ccc} & | & | & | \\ I_{n-3} & v_2+v_3 & v_1+v_2+v_3 & v_1+v_3 \\ & | & | & | \\ \hline & 0 & 1 & 1 \\ 0_{(n-3)\times 3} & 1 & 1 & 1 \\ & 1 & 0 & 1 \end{array}\right].$$

In Cases 4(a) and 4(b), this may be adjusted to the following standard forms

$$4.(\text{a}) \left[\begin{array}{c|ccc} & | & | & | \\ I_{n-3} & v_2+v_3 & v_1+v_3 & v_1+v_2+v_3 \\ & | & | & | \\ \hline & 1 & 0 & 1 \\ 0_{(n-3)\times 3} & 1 & 1 & 1 \\ & 0 & 1 & 1 \end{array}\right]$$

$$4.(\text{b}) \left[\begin{array}{c|ccc} & | & | & | \\ I_{n-3} & v_1+v_3 & v_2+v_3 & v_1+v_2+v_3 \\ & | & | & | \\ \hline & 1 & 0 & 1 \\ 0_{(n-3)\times 3} & 1 & 1 & 1 \\ & 0 & 1 & 1 \end{array}\right]$$

The matrices above are again of types 4(b) and 4(a), respectively; the graphs that represent 4(a) and 4(b) are equivalent.

5. In Case 5,

$$P^{-1} = \left[\begin{array}{c|ccc} & | & | & | \\ I_{n-3} & v_1 & v_2 & v_1+v_3 \\ & | & | & | \\ \hline & 1 & 0 & 1 \\ 0_{(n-3)\times 3} & 0 & 1 & 0 \\ & 0 & 0 & 1 \end{array}\right].$$

In Cases 5(a) and 5(b), this may be adjusted to the following standard forms

$$5.(\text{a}) \left[\begin{array}{c|ccc} & | & | & | \\ I_{n-3} & v_2 & v_1 & v_1+v_3 \\ & | & | & | \\ \hline & 1 & 0 & 0 \\ 0_{(n-3)\times 3} & 0 & 1 & 1 \\ & 0 & 0 & 1 \end{array}\right] \quad 5.(\text{b}) \left[\begin{array}{c|ccc} & | & | & | \\ I_{n-3} & v_2 & v_1 & v_1+v_3 \\ & | & | & | \\ \hline & 1 & 0 & 0 \\ 0_{(n-3)\times 3} & 0 & 1 & 1 \\ & 0 & 0 & 1 \end{array}\right]$$

The matrices above are again of types 6(b) and 6(a), respectively, and we conclude that Cases 5(a) and 5(b) are respectively equivalent to 6(b) and 6(a), in terms of the covering groups that they describe.

If $n \geq 7$, a 2-uniform covering group of corank 3 of C_2^n that satisfies the conditions of Lemma 5.1 possesses exactly two 2-uniform bases $\mathcal{B}$ and $\mathcal{B}'$, up to coset representatives modulo G'. The graphs corresponding to the two bases are encoded by the change of basis matrices P and P^{-1}, and are typically non-isomorphic. The conclusion of this section is that in order to list all isomorphism types of such groups, it is sufficient to consider matrices of types 1(a), 1(b), 2(a), 2(b), 3(a), 3(b), 4(a), 5(a) and 5(b). The associated graphs capture every group isomorphism type once, except for those encoded by matrices of types 2(a) and 2(b), which are generally represented by two different graphs. Since the three columns in the upper right $(n-3)\times 3$ region can be chosen independently, the number of matrices of each of these types is $(2^{n-4})^3$. Most isomorphism types of groups of types 2(a) and 2(b) are counted twice by this count of distinct matrices, but in all other cases, the distinct matrices correspond bijectively with the isomorphism classes of groups. The number of isomorphism types of 2-uniform covering groups of C_2^n and uniform corank 3, that admit two different choices for the common square of exactly three elements of a 2-uniform basis, is approximately

$$8\times(2^{n-4})^3 = 2^{3n-9}.$$

The qualifier "approximately" refers to the few cases in which the two 2-uniform bases of a covering group of type 2 determine isomorphic 2-colored graphs.

6. Groups of uniform corank 2

Let G be a 2-uniform covering group of C_2^n of uniform corank 2, where $n\geq 6$. Let $\mathcal{B}=\{x_1,\ldots,x_k,y_1,y_2\}$ be a 2-uniform basis of G, where $x_i^2=r$ and $y_i^2=s$, $r\neq s$. By Theorem 3.5, no element of G', apart from r and possibly s, is the square of more than three independent elements of G', but it is possible that y_1 and y_2 can be replaced in $\mathcal{B}$ by elements z_1 and z_2, to form an alternative 2-uniform basis $\mathcal{B}'$. In this situation, $\mathcal{B}'=\{x_1,\ldots,x_k,z_1,z_2\}$, where $z_1^2=z_2^2=s'$ and $s'\notin\{r,s\}$. We now consider the conditions on $\Gamma_{\mathcal{B}}(G)$ which admit this possibility, and discuss the relationship between the graphs $\Gamma_{\mathcal{B}}(G)$ and $\Gamma_{\mathcal{B}'}(G)$.

We assume that G contains elements z_1 and z_2 as described above, and as in Section 5 we write $\mathcal{X}$ and $\mathcal{Y}$ for the subsets $\{x_1,\ldots,x_k\}$ and $\{y_1,y_2\}$ of $\mathcal{B}$. We may assume that each of z_1 and z_2 is a product of elements of $\mathcal{B}$, and we write $\mathcal{Z}_1$ and $\mathcal{Z}_2$ respectively for the sets of elements of $\mathcal{B}$ that occur in z_1 and z_2. We note that each of $\mathcal{Z}_1$ and $\mathcal{Z}_2$ has at least two elements. That $\mathcal{X}\cup\{z_1,z_2\}$ generates G requires that the sets $\mathcal{Z}_1\cap\mathcal{Y}$ and $\mathcal{Z}_2\cap\mathcal{Y}$ are distinct and non-empty. Comparing the expressions for z_1^2 and z_2^2 in terms of the elements of $\mathcal{B}$, we observe that $z_1^2=z_2^2$ if and only if one of the following conditions holds.

Case 1 $r=C_1C_2$, where C_1 and C_2 are elements of G' represented with respect to $\mathcal{B}$ by cliques on the sets of vertices corresponding to $\mathcal{Z}_1$ and $\mathcal{Z}_2$ respectively. This occurs if $|\mathcal{X}\cap\mathcal{Z}_1|$ and $|\mathcal{X}\cap\mathcal{Z}_2|$ have opposite parity, and $|\mathcal{Y}\cap\mathcal{Z}_1|$ and $|\mathcal{Y}\cap\mathcal{Z}_2|$ have the same parity (which must be odd). After relabeling, we may interpret this last condition as saying that $y_1\in\mathcal{Z}_1\backslash\mathcal{Z}_2$, $y_2\in\mathcal{Z}_2\backslash\mathcal{Z}_1$, $|\mathcal{Z}_1|$ is odd and $|\mathcal{Z}_2|$ is even.

Case 2 $s=C_1C_2$, where C_1 and C_2 are elements of G' represented with respect to $\mathcal{B}$ by cliques on the sets of vertices corresponding to $\mathcal{Z}_1$ and $\mathcal{Z}_2$ respectively. This occurs if $|\mathcal{X}\cap\mathcal{Z}_1|$ and $|\mathcal{X}\cap\mathcal{Z}_2|$ have the same parity, and $|\mathcal{Y}\cap\mathcal{Z}_1|$ and $|\mathcal{Y}\cap\mathcal{Z}_2|$ have opposite parity. After relabeling, we may infer from this last condition that $y_1\in\mathcal{Z}_1\cap\mathcal{Z}_2$, and $y_2\in\mathcal{Z}_2\backslash\mathcal{Z}_1$. We distinguish the following subcases:
Case 2(a) $|\mathcal{X}\cap\mathcal{Z}_1|$ and $|\mathcal{X}\cap\mathcal{Z}_2|$ are odd.
Case 2(b) $|\mathcal{X}\cap\mathcal{Z}_1|$ and $|\mathcal{X}\cap\mathcal{Z}_2|$ are even.

Case 3 $rs=C_1C_2$, where C_1 and C_2 are elements of G' represented with respect to $\mathcal{B}$ by cliques on the sets of vertices corresponding to $\mathcal{Z}_1$ and $\mathcal{Z}_2$ respectively. This occurs if $|\mathcal{X}\cap\mathcal{Z}_1|$ and $|\mathcal{X}\cap\mathcal{Z}_2|$ have opposite parity, and $|\mathcal{Y}\cap\mathcal{Z}_1|$ and $|\mathcal{Y}\cap\mathcal{Z}_2|$ have opposite parity. As in the second case above, we may assume in this situation that $y_1\in\mathcal{Z}_1\cap\mathcal{Z}_2$, and $y_2\in\mathcal{Z}_2\backslash\mathcal{Z}_1$. Again we consider two subcases, depending on the numbers of blue vertices in the cliques describing C_1 and C_2.
Case 3(a) $|\mathcal{X}\cap\mathcal{Z}_1|$ is odd and $|\mathcal{X}\cap\mathcal{Z}_2|$ is even.
Case 3(b) $|\mathcal{X}\cap\mathcal{Z}_1|$ is even and $|\mathcal{X}\cap\mathcal{Z}_2|$ is odd.

It is possible for more than one of Cases 1, 2 and 3 to occur simultaneously, so that there may be multiple choices for the pair of elements $\{z_1,z_2\}$. It is even possible, in Case 3(b), that the same graph may admit two different choices for C_1 and C_2, in a case where rs is represented by a 4-cycle that has two different descriptions as the symmetric difference of two copies of the complete graph K_3. In all other cases, it follows from Theorem 3.5 and the parity restrictions that there is only one possible choice for the pair (C_1, C_2) corresponding to the description of r, s or rs as a product of two elements represented by complete graphs.

In each of the three cases, we write $\mathcal{B}'$ for the basis obtained from $\mathcal{B}$ by replacing y_1 and y_2 by z_1 and z_2, and consider the relationship between the graphs $\Gamma_{\mathcal{B}}$ and $\Gamma_{\mathcal{B}'}$. We consider these two graphs to have the same vertex set, where the red vertices that represent y_1 and y_2 in $\Gamma_{\mathcal{B}}$ respectively represent z_1 and z_2 in $\Gamma_{\mathcal{B}'}$. In all cases, Theorem 2.9 provides a template for the description of the relationship between the two graphs.

As in Section 5, we may consider the change of basis matrix P from $\mathcal{B}'$ to $\mathcal{B}$, whose columns list the coordinates of the elements of $\mathcal{B}'$ with respect to $\mathcal{B}$. Unlike the case of uniform corank 3, this matrix does not fully describe the group, but only one of the three elements r, s and rs. The matrix P, and its inverse, have the following forms.

$$P = \left[\begin{array}{c|cc} & | & | \\ I_{n-2} & v_1 & v_2 \\ & | & | \\ \hline & 1 & e \\ 0_{(n-2)\times 2} & 0 & 1 \end{array}\right], \qquad P^{-1} = \left[\begin{array}{c|cc} & | & | \\ I_{n-2} & v_1 & ev_1 + v_2 \\ & | & | \\ \hline & 1 & e \\ 0_{(n-2)\times 2} & 0 & 1 \end{array}\right],$$

where $e = 0$ or 1, and v_1 and v_2 are columns with entries in $\mathbb{F}_2$. We write $n(v)$ for the number of non-zero entries in the column vector v. If $e = 0$, then $n(v_1)$ is even and $n(v_2)$ is odd. The condition that $z_1^2 = z_2^2$ means that the above cases and subcases are encoded in the matrix P as in the following table.

	e	$n(v_1)$	$n(v_2)$
Case 1	0	even	odd
Case 2(a)	1	odd	odd
Case 2(b)	1	even	even
Case 3(a)	1	odd	even
Case 3(b)	1	even	odd

From the description of P^{-1} in terms of P, we note that if P describes an instance of Case 2(a), then P^{-1} describes one of Case 3(a), and vice versa. Hence every 2-uniform graph that satisfies condition 2(a) is equivalent to one that satisfies condition 3(b), and it is sufficient to consider one of these conditions in a description of graphs that describe 2-uniform covering groups of uniform corank 2, up to isomorphism.

In all other rows of the table above, the matrices P and P^{-1} correspond to the same row of the table. In these cases, the relationship between the 2-uniform graphs $\Gamma_{\mathcal{B}}(G)$ and $\Gamma_{\mathcal{B}'}(G)$ is described by Theorem 2.9.

1. In Case 1, we have $s' = z_1^2 = sC_1$. By Corollary 2.8, the graphs representing C_1 and C_2, and hence r, are the same with respect to both bases, so $\Gamma_{\mathcal{B}}(G)$ and $\Gamma_{\mathcal{B}'}(G)$ have the same sets of blue edges. We write Q_1 and Q_2 for the respective sets of blue vertices in the cliques representing the elements C_1 and C_2 with respect to $\mathcal{B}$, and we write P_1 and P_2 for the sets of neighbors of the vertices representing y_1 and y_2 in $\Gamma_{\mathcal{B}}(s)$. Then the set of red edges of $\Gamma_{\mathcal{B}'}(s')$, hence of $\Gamma_{\mathcal{B}'}(G)$, is given by $E(\Gamma_{\mathcal{B}'}(s)) \triangle E(C_1)$, and from Theorem 2.9 we have
$$E(\Gamma_{\mathcal{B}'}(s)) = \begin{cases} E(\Gamma_{\mathcal{B}}(s)) \triangle E(P_1, Q_1) \triangle E(P_2 \triangle Q_1, Q_2) & \text{if the red vertices of } \Gamma_{\mathcal{B}}(G) \text{ are adjacent via a red edge} \\ E(\Gamma_{\mathcal{B}}(s)) \triangle E(P_1, Q_1) \triangle E(P_2, Q_2) & \text{otherwise} \end{cases}$$
2. In Case 2(b), $s' = z_2^2 = C_2$. By inspecting the entries of the last two columns of P^{-1} (or by applying Theorem 2.9 to the graph of C_2 with respect to $\mathcal{B}$), we observe that the set of red edges in $\Gamma_{\mathcal{B}'}(G)$ is the symmetric difference of the edge sets of the cliques on the sets of vertices representing $\mathcal{Z}_1$ and $(\mathcal{Z}_1 \triangle \mathcal{Z}_2) \cup \{z_1\}$. The blue edges of $\Gamma_{\mathcal{B}}(G)$ are independent of the red edges and of condition 2(a), and Theorem 2.9 describes how they change under the change of basis. We write P_1 and P_2 for the sets of neighbors in $\Gamma_{\mathcal{B}}(r)$ of the vertices representing y_1 and y_2 respectively, and Q_1 and Q_2 for the

sets of vertices respectively representing $\mathcal{Z}_1\backslash\{y_1\}$ and $\mathcal{Z}_2\backslash\{y_2\}$. Then the set of blue edges of $\Gamma_{\mathcal{B}'}(G)$ is given by

$$E(\Gamma_{\mathcal{B}'}(r)) = \begin{cases} E(\Gamma_{\mathcal{B}}(r))\triangle E(P_1,Q_1)\triangle E(P_2\triangle Q_1,Q_2) & \text{if the red vertices of } \Gamma_{\mathcal{B}}(G) \text{ are adjacent via a blue edge} \\ E(\Gamma_{\mathcal{B}}(r))\triangle E(P_1,Q_1)\triangle E(P_2,Q_2) & \text{otherwise} \end{cases}$$

3. In Case 3(a), $s' = z_1^2 = sC_1$. From the matrix P^{-1} we note that $\Gamma_{\mathcal{B}'}(rs')$ is the symmetric difference of the cliques on the sets of vertices representing $\mathcal{Z}_1$ and $(\mathcal{Z}_1\triangle\mathcal{Z}_2)\cup\{z_1\}$, which respectively involve an even and odd number of blue vertices. The set of blue edges in $\Gamma_{\mathcal{B}'}(G)$ is given, as in Case 2(b), by

$$E(\Gamma_{\mathcal{B}'}(r)) = \begin{cases} E(\Gamma_{\mathcal{B}}(r))\triangle E(P_1,Q_1)\triangle E(P_2\triangle Q_1,Q_2) & \text{if the red vertices of } \Gamma_{\mathcal{B}}(G) \text{ are adjacent via a blue edge} \\ E(\Gamma_{\mathcal{B}}(r))\triangle E(P_1,Q_1)\triangle E(P_2,Q_2) & \text{otherwise} \end{cases},$$

where P_1, P_2, Q_1, Q_2 have the same definitions as in Case 2(b). Finally, the set of red edges in $\Gamma_{\mathcal{B}'}(G)$ is the symmetric difference of the edge sets of the graphs representing rs' and r.

7. Groups of uniform corank 1

Throughout this section, we suppose that G is a 2-uniform covering group of C_2^n, with $\rho(G) = n-1$, where $n \geq 5$. Let $x_1,\ldots,x_{n-1}$ be independent elements of G, all with square r. Then $\{x_1,\ldots,x_{n-1}\}$ may be extended to a 2-uniform basis of G by the addition of *any* element y of G that does not belong to the subgroup $X = \langle x_1,\ldots,x_{n-1}\rangle$. Every 2-uniform basis includes $n-1$ elements with square r, by Theorem 3.5. Having chosen y, we write $\Gamma(y)$ for the graph of G with respect to the basis $\{x_1,\ldots,x_{n-1},y\}$, which has $n-1$ blue vertices representing $x_1,\ldots,x_{n-1}$, and a single red vertex representing y.

Lemma 7.1. *The neighbors in $\Gamma(y)$ of the red vertex, via blue edges, do not depend on the choice of y.*

Proof. Suppose that y and y' are different elements of $G\backslash X$. Then $y \in y'xG'$ for some $x \in X$. After relabeling the elements of $\mathcal{B}$ we may suppose that

$$r = [y, x_1\ldots x_p]c,$$

where $c \in X'$. Then

$$r = [y'x, x_1\ldots x_p]c = [y', x_1\ldots x_p]c',$$

where $c' \in X'$. Hence the neighbors of the red vertex in the blue parts of both $\Gamma(y)$ and $\Gamma(y')$ are the vertices representing $x_1,\ldots,x_p$. □

We continue to write $\{x_1,\ldots,x_p\}$ for the set of neighbors of the red vertex via blue edges, in a 2-uniform graph representing G.

Lemma 7.2. *If p is even, then for every subset S of $\{x_1,\ldots,x_{n-1}\}$, there is exactly one choice of y for which the red vertex is adjacent via red edges in $\Gamma(y)$ precisely to those vertices representing elements of S. In particular there is exactly one choice of y for which the red vertex is incident with no red edge in $\Gamma(y)$.*

Proof. We assume that p is even, and choose $z \in G\backslash X$. If $x_{j_1},\ldots,x_{j_q}$ are the basis elements representing the neighbors of the red vertex via red edges in $\Gamma(z)$, we may write

$$z^2 = [z, x_{j_1}\ldots x_{j_q}]\,c,$$

where $c \in X'$. Define y by

$$y = \begin{cases} zx_{j_1}\ldots x_{j_q} & \text{if } q \text{ is even} \\ zx_{j_1}\ldots x_{j_q}x_1\ldots x_p & \text{if } q \text{ is odd} \end{cases}$$

For even q, $y^2 = z^2 r^q C(z, x_{j_1}, \ldots, x_{j_q}) \in [z, x_{j_1} \ldots x_{j_q}]^2 X'$, and the red vertex in $\Gamma(y)$ is incident with no red edge.
For odd q, $y^2 = z^2 r^q [z, x_{j_1} \ldots x_{j_q} x_1 \ldots x_p] C(\{x_{j_1} \ldots, x_{j_q}\} \triangle \{x_1, \ldots, x_p\})$. Since $r^q = r \in [z, x_1 \ldots x_p]X'$, again in this case we have $y^2 \in X'$, and the red vertex in $\Gamma(y)$ is incident with no red edge.

For any subset $S = \{x_{i_1}, \ldots, x_{i_t}\}$ of $\{x_1, \ldots, x_{n-1}\}$, we may define y_S by

$$y_S = \begin{cases} yx_{i_1} \ldots x_{i_t} & \text{if } t \text{ is even} \\ yx_{i_1} \ldots x_{i_t} x_1 \ldots x_p & \text{if } t \text{ is odd} \end{cases}$$

Then it is easily confirmed that the neighbors via red edges of the red vertex in $\Gamma(y_S)$ are exactly those blue vertices that represent elements of S. Moreover every possible neighbor set occurs for exactly one choice of an element of G/G' that completes $\{x_1 G', \ldots, x_{n-1} G'\}$ to a basis of G/G'. □

The following lemma deals with the alternative case, where the red vertex is adjacent via blue edges to an odd number of blue vertices.

Lemma 7.3. *If p is odd, then the red degree of the red vertex is either even for every choice of y or odd for every choice of y. Furthermore,*

1. *If this degree is even for all y, then for every subset S of even cardinality of $\{x_1, \ldots, x_{n-1}\}$, there are exactly two choices of yG' for which the neighbors via red edges of the red vertex in $\Gamma(y)$ are precisely those vertices representing elements of S. These two choices of y differ from each other (modulo G') by the element $x_1 \ldots x_p$, the product of the basis elements represented by the neighbors of the red vertex via blue edges. In particular, in this case there are two choices of yG' for which the red vertex is incident with no red edge in $\Gamma(y)$.*
2. *If this degree is odd for all y, then for every subset S of odd cardinality of $\{x_1, \ldots, x_{n-1}\}$ there are exactly two choices of yG' for which the neighbors via red edges of the red vertex in $\Gamma(y)$ are precisely those vertices representing elements of S. These two choices of y differ from each other (modulo G') by the element $x_1 \ldots x_p$. In particular, there are two choices of yG' for which the red vertex in $\Gamma(y)$ has the same neighbor set via red and blue edges.*

Proof. We assume that p is odd and choose $z \in G \backslash X$. We write

$$z' = zx_1 \ldots x_p.$$

Then

$$\begin{aligned} (z')^2 &= z^2 r^p C(z, x_1, \ldots, x_p) \\ &= [z, x_1 \ldots x_p] r [z, x_1 \ldots x_p] c, \\ &= rc. \end{aligned}$$

where $c \in X'$. Thus the red vertex has the same set of neighbors in $\Gamma(z)$ and $\Gamma(z')$, whenever z' and z are related by $z' \in zx_1 \ldots x_p G'$.

Now let S be any subset of $x_1, \ldots, x_{n-1}$ and let x be the product of the elements of S (in some order). Choose $y \in G \backslash \langle X \rangle$, and let N_y be the set of neighbors of the red vertex via red edges in $\Gamma(y)$. Then

$$(yx)^2 \in y^2 r^{|S|} [yx, x] X'.$$

Thus the set of neighbors via red edges of the red vertex in $\Gamma(yx)$ is
- $N_y \triangle S$, if $|S|$ is even;
- $N_y \triangle S \triangle \{x_1, \ldots, x_p\}$, if $|S|$ is odd.

Since p is odd, the red degree of the red vertex has the same parity in $\Gamma(y)$ and $\Gamma(yx)$, for all choices of x. Since the symmetric difference is a group operation on the power set of $\{x_1, \ldots, x_{n-1}\}$, every subset whose cardinality has the same parity as N_y occurs (as the neighbor set via red edges of the red vertex) for two choices of S, one with odd and one with even cardinality.

In particular, if $|N_y|$ is even, then $\{x_1, \ldots, x_{n-1}\}$ may be extended (in two ways) to a 2-uniform basis of G whose graph has the property that its red vertex is incident with no red edge. If $|N_y|$ is odd, the $\{x_1, \ldots, x_{n-1}\}$ may be extended (in two ways) to a 2-uniform basis whose graph has the property that the neighbors of the red vertex via red edges coincide with those via blue edges. □

It remains to consider the relationship between the two 2-uniform graphs representing G, and having the properties described in Lemma 7.3, in the case that p is odd. Suppose that G is a group satisfying the hypotheses of Lemma 7.3, and that the element y of $G\backslash X$ has been chosen so that the red vertex in graph $\Gamma(y)$ is either incident with no red edge, or has the same set of p neighbors via both blue and red edges. Then we have the following lemma.

Lemma 7.4. *Let $x_1, \ldots, x_p$ be the basis elements represented by the neighbors of the red vertex, via blue edges, in $\Gamma(y)$, where p is odd. Let $y' = x_1 \ldots x_p y$. Then the graph $\Gamma(y')$ that represents G with respect to the basis $\{x_1, \ldots x_{n-1}, y'\}$ is related to $\Gamma(y)$ as follows:*

- *The two graphs are considered to have the same vertex set, where the red vertex represents y in $\Gamma(y)$ and y' in $\Gamma(y')$;*
- *$\Gamma(y)$ and $\Gamma(y')$ have the same set of blue edges;*
- *The set of red edges in $\Gamma(y')$ is given by $E^R(\Gamma(y)) \triangle S \triangle T$, where S and T respectively denote the set of blue edges amongst the blue vertices of $\Gamma(y)$ and the edge set of the complete graph on the vertices representing $x_1, \ldots, x_p$.*

Proof. That the sets of blue vertices coincide in $\Gamma(y)$ and $\Gamma(y')$ follows from the fact that

$$r = [y, x_1 \ldots x_p]C = [y', x_1 \ldots x_p]C,$$

where C is a product of commutators involving the elements $x_1, \ldots, x_{n-1}$, which is represented by the same set of edges in both graphs.

That the sets of red edges are related as described above follows from the observation that

$$\begin{aligned}(y')^2 &= (x_1 \ldots x_p y)^2 \\ &= r^p s[x_1 \ldots x_p, y] \prod_{1 \le j < k \le p} [x_j, x_k] \\ &= rs[x_1 \ldots x_p, y'] \prod_{1 \le j < k \le p} [x_j, x_k].\end{aligned}$$

The red edges of $\Gamma(y')$ are those that represent commutators that occur in the element $s' = (y')^2$ of G', with respect to the basis $\{x_1, \ldots, x_{n-1}, y'\}$. For $1 \le j \le p$, the commutators $[x_j, y']$ all occur in r, and either all or none of them occur in s. Hence they occur in s' if and only if they occur in s, and the sets of red edges incident with the red vertex coincide in $\Gamma(y)$ and $\Gamma(y')$. For basis elements x_i and x_j represented by blue vertices, the commutator $[x_i, x_j]$ occurs in s' if and only if it occurs in exactly one of s, r and $\prod_{1 \le j < k \le p} [x_j, x_k]$ or in all three of them, hence the conclusion. □

In order to classify 2-uniform covering groups of C_2^n of uniform corank 1 with 2-uniform graphs, it is sufficient to consider 2-uniform graphs with a single red vertex, which is either incident with no red edge, or has the same set of neighbors, of odd cardinality, via both red and blue edges. We refer to such graphs as being in *standard form*. Such a graph could admit a simple exchange operation of Type 1 in Theorem 4.2, if its blue edges form a clique on an even number of blue vertices. If $n \ge 5$, no other graph transformations can arise, that preserve the property of being in standard form and the isomorphism type of the group.

Assume that $n \ge 5$ and let G be a covering group of C_2^n of uniform corank 1. If G is represented by a 2-uniform graph in standard form, in which the red vertex is incident with a positive even number of blue edges, then this is the only example in standard form that represents G. If G is represented by a 2-uniform graph in standard form where the red vertex is isolated, then this is the only graph in standard form that represents G, unless it admits exchange operations as mentioned above. If the red vertex is incident with an odd number of blue edges in a standard 2-uniform graph representing Γ, then it follows from Theorem 4.2 that no exchange operations preserving the property of being in standard form are possible. However, each group of this type is represented by two generally non-isomorphic graphs in standard form, as described in Lemma 7.4. The collection of all standard 2-uniform graphs in which the red vertex is incident with an odd number of blue edges has two graphs representing each of the groups that occur, with exceptions only in cases where the two graphs described in Theorem 7.4 are isomorphic. Graphs in this collection have a natural occurrence in pairs; corresponding to each example in which the red vertex is incident with no red edge, is one in which the red vertex has the same neighbors via red edges as blue. Those graphs in which the red vertex is incident with no red edge account for half of all graphs in this collection, and their number approximates (and slightly overestimates) the number of isomorphism types of covering groups involved. We conclude that for $n \geq 5$, the number of isomorphism types of covering groups of uniform corank 1 of C_2^n is closely approximated by the number of 2-uniform graphs of standard form on n vertices, in which the red vertex is incident with no red edge.

8. Conclusion

A goal of this article was to identify a class of 2-colored graphs of order n, whose graph isomorphism types encode the isomorphism types of 2-uniform covering groups of the elementary abelian group C_2^n. A description of such a class would establish a 2-uniform analogue of Theorem 2.4, which states that the isomophism types of uniform covering groups of C_2^n are in bijective correspondence with the isomorphism types of simple undirected graphs on n vertices. The proof of this theorem in [3] amounts to the observation that a uniform covering group has a unique uniform basis, except in a few special cases, where the alternative uniform bases determine isomorphic graphs. For 2-uniform graphs, there are much more extensive conditions admitting the existence of multiple 2-uniform bases in a particular group. When the uniform corank exceeds 3, the description of 2-uniform graphs in Theorem 3.10 is an approximate analogue of Theorem 2.4. It provides a correspondence that fails to be bijective only in the few special cases detailed in Section 4. In the exceptional cases where multiple non-isomorphic graphs are equivalent in the sense of representing the same group, we have not found a systematic way to refine the correspondence by selecting a single representative of each equivalence class. This situation is exacerbated in the case of uniform corank 2 and 3, due to a wider range of configurations in which multiple 2-uniform bases occur.

The case of 2-uniform graphs of uniform corank 1 is special, because of the possibility to restrict attention to bases, and graphs, of standard form. As outlined in the conclusion of Section 7, the results of this section most closely resemble Theorem 2.4. The cases of 2-uniform graph of uniform rank at most 3 remain to be considered, and will be the subject of another article.

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Disclosure statement

The authors declare that there is no competing interest associated with this work.

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