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Bernstein polynomials method for solving multi-order fractional neutral pantograph equations with error and stability analysis

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ABSTRACT

In this investigation, we present a new method for addressing fractional neutral pantograph problems, utilizing the Bernstein polynomials method. We obtain solutions for the fractional pantograph equations by employing operational matrices of differentiation, derived from fractional derivatives in the Caputo sense applied to Bernstein polynomials. Error analysis, along with Chebyshev algorithms and interpolation nodes, is employed for solution characterization. Both theoretical and practical stability analyses of the method are provided. Demonstrative examples indicate that our proposed techniques occasionally yield exact solutions. We compare the algorithms using several established analytical methods. Our results reveal that our algorithm, based on Bernstein series solution methods, outperforms others, exhibiting superior performance with higher accuracy orders compared to those obtained from Chebyshev spectral methods, Bernoulli wavelet method, and Spectral Tau method.

1. Introduction

The exploration of various phenomena through fractional differential equations has become increasingly significant. This field has deep historical roots, stretching back to 1695 when L'Hopital engaged Leibniz in a discourse to ascertain the importance of $\frac{d^n y}{dx^n}$ for the order's derivative when $n = \frac{1}{2}$. In recent times, fractional differential equations have found applications across various domains of mathematical engineering and physics, particularly in fields such as fluid mechanics [1], physical modeling [2], medicine [3], and signal processing engineering [4], see, [5,6]. One recent cutting-edge study delved into the significant application of fractional differential equations in modeling anomalous diffusion processes. For example, in paper [7], the authors put forward a novel numerical approach on the basis of Euler wavelet for resolving fractional diffusion-wave equations. In [8], the authors explored the existence and uniqueness of iterative fractional differential equations, employing the fixed-point theorem under specific conditions. Moreover, a new operational matrix based on the Bernstein matrices method, employing integration, was proposed to tackle a category of fractional integral differential equations [9].

A delay differential equation is characterized by a differential equation where the state variable is dependent on delayed arguments [10]. Fractional delay differential equations (FDDEs) denote equations that incorporate both fractional derivatives and time delays. These equations find applications in diverse fields of applied sciences including physics, biology, economics, bioengineering, and hydraulic networks [11]. FDDEs have been utilized in bioengineering to elucidate the dynamics occurring within biological tissues [12]. In recent years, researchers have continued their investigation into FDDEs, exploring their properties and applications across various fields. In [13], the author resolved a fractional delay differential equation utilizing Fraction Taylor's series through the traditional reproducing kernel method. Furthermore, a Fractional Legendre polynomials method was employed to address a specific category of fractional delay differential equations, accompanied by convergence analysis [14].

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Among the significant fractional delay equations, pantograph equations stand out prominently. These equations are frequently employed to model various real-life phenomena. Analytical methods have been utilized extensively to derive closed-form solutions for fractional pantograph equations. In [15], the spectral Tau method was leveraged to resolve multi-pantograph equation systems. Chebyshev spectral methods have been applied in order to address multi-order fractional neutral pantograph equations [16]. In the recent past, ψ Caputo fractional derivatives were implemented to evaluate the solution pertaining to fractional pantograph problems associated with boundary conditions. It is noteworthy that this study considered scenarios where $\psi(t) = t$ refer to [17]. The study explores the existence, uniqueness, and diverse forms of Ulam–Hyers (UH)-type stability outcomes for nonlocal pantograph equations featuring psi-piecewise Caputo fractional derivatives [18].

Recently, several methods reliant on the collocation technique have emerged for solving fractional differential equations. Among these, the Bernstein polynomial method holds significant importance. Central to this method is the translation of each term in the problem into matrix form. Fractional Bernstein matrices extend the concept of standard Bernstein polynomials by expressing $t \to t^{\alpha}$. Numerous researchers have employed the Bernstein polynomials method with a view to addressing significant impediments in applied mathematics and physics; refer to [19–22].

FDDEs find widespread applications across engineering, applied science, and various other fields. It is noteworthy to emphasize that a comprehensive examination of fractional calculus has been conducted to delineate the diverse applications involving FDDEs. In this paper, we delve into the following two fractional delay differential equations:

Problem 1.

$$\begin{cases} D^{(\gamma)}u(t) = f(t, u(t), u(\delta t - \omega)), 0 \le t \le 1, \quad n-1 < \gamma \le n \\ u^{(i)}(0) = \lambda_i \quad i = 0, 1, \dots, n-1. \end{cases} \tag{1}$$

Problem 2 (*Fractional Multi-Pantograph System*)**.**

$$\begin{aligned} D^{(\gamma)}u_1(t) &= \alpha_1 u_1(t) + f_1(t, u_i(t), u_i(\delta_j t)) \\ D^{(\gamma)}u_2(t) &= \alpha_2 u_1(t) + f_2(t, u_i(t), u_i(\delta_j t)) \\ &\vdots \\ D^{(\gamma)}u_n(t) &= \alpha_n u_n(t) + f_n(t, u_i(t), u_i(\delta_j t)) \\ u^{(i)}(0) &= \lambda_i, \quad i, j = \in 0, 1, \dots, n. \end{aligned} \tag{2}$$

Leveraging the matrix relations between the Bernstein matrices $B_n(t)$ and their derivatives, we present a numerical method to solve Eqs. (1) and (2). By implementing the Caputo sense, Bernstein operational matrix of derivative $\Omega(t)$ is introduced. To illustrate the effectiveness of the proposed techniques, we construct differentiation operational matrices for a problem with a non-smooth exact solution. We then compare these solutions with several established approaches, including the Bernoulli wavelets method, the Chebyshev spectral method, the Spectral Tau method, the fractional Adams method (FAM), and the new predictor–corrector method (NPCM).

This paper is organized as follows: In Section 2, we provide essential background on fractional calculus and define Bernstein polynomials. In Section 3, we introduce a numerical algorithm, the Bernstein series solution (BSS), utilizing operational matrices of differentiation (referred to as BSSD), to solve the proposed FDDEs. The solution is obtained by applying the conditions and employing the Gauss elimination procedure. In Section 4, we outline the residual correction procedure to estimate the absolute error and assess the stability of the methods. Section 5 presents several numerical experiments to validate the approach for various values of n. Finally, Section 6 concludes the paper.

2. Preliminaries and notations

2.1. Fractional derivatives

This section presents the definitions and properties of fractional calculus [23].

Definition 2.1. The Riemann–Liouville fractional integral operator (J^{γ}) of order $\gamma \ge 0$, of a function $u \in C_{\mu}$, $\mu \ge -1$, is defined as follows:

$$\begin{aligned} J^{\gamma}u(t) &= \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} u(t) ds \quad (\gamma > 0), \\ J^0 u(t) &= u(t), \end{aligned} \tag{3}$$

where $\Gamma(\gamma)$ denotes a well-known gamma function. Certain properties of the operator J^{γ}, which will be required her, are: For $u \in C_{\mu}$, $\mu \ge -1$, $\gamma, \beta \ge 0$ and $\gamma \ge -1$:

1. $J^{\gamma} J^{\beta} u(t) = J^{\gamma+\beta} u(t)$,
2. $J^{\gamma} J^{\beta} u(t) = J^{\beta} J^{\gamma} u(t)$,
3. $J^{\gamma} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\gamma+\beta+1)} t^{\gamma+\beta}$.

Definition 2.2. In Caputo's sense, the fractional derivative (D^{γ}) of $u(t)$ is defined in the following manner:

$$D^{\gamma}u(t) = \frac{1}{\Gamma(n-\gamma)}\int_0^t (t-s)^{n-\gamma-1}u^{(n)}(t)ds, \tag{4}$$

for $n-1<\gamma<n, n\in N, t>0, u\in C_{-1}^n$.

The two rudimentary properties of the Caputo fractional derivative are as follows: [24]:

1. Let $u\in C_{-1}^n, n\in N$. Than $D^{\gamma}u, 0\le\gamma\le n$ is well defined and $D^{\gamma}u\in C_{-1}$.
2. Let $n-1\le\gamma\le n, n\in N$ and $u\in C_{\mu}^n, \mu\ge -1$. Then

$$(J^{\gamma}D^{\gamma})u(t) = u(t) - \sum_{k=0}^{n-1} u^{(k)}(0^+)\frac{t^k}{k!}. \tag{5}$$

With regard to the Caputo derivative, we have

$$D_*^{\gamma}c = 0, \quad (c \quad \text{constant}), \tag{6}$$

$$D_*^{\gamma}t^{\beta} = \begin{cases} 0, & \text{for} \quad \beta\in N_0 \quad \text{and} \quad \gamma<\lceil\gamma\rceil, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\gamma)}t^{\beta-\gamma}, & \text{for} \quad \gamma\in N_0 \quad \text{and} \quad \beta\ge\lceil\gamma\rceil \quad \text{or } \gamma>\lfloor\gamma\rfloor. \end{cases} \tag{7}$$

Notably, this study will utilize the Caputo fractional derivative along with its properties to identify the approximate solutions.

2.2. Definitions of Bernstein polynomials

The following equation elucidates the nth degree Bernstein polynomials:

$$B_{r,n}(t) = \binom{n}{r}\frac{t^r(R-t)^{n-r}}{R^n}, \quad r=0,1,2,\dots,n \quad t\in[0,R], \tag{8}$$

In a similar vein, $B_{r,n}^{\gamma}(t)$ fractional Bernstein polynomials are formed by means of $x\to x^{\gamma}$, which is why Eq. (8) becomes

$$B_{r,n}^{\gamma}(t) = \binom{n}{r}\frac{t^{r\gamma}(R-t^{\gamma})^{n-r}}{R^n}, \quad 0<\gamma<1, \qquad r=0,1,2,\dots,n \quad t\in[0,R], \tag{9}$$

In addition, a recursive definition is capable of producing the B-polynomials over this interval; thus, the nth degree of B-polynomials can be written as follows:

$$B_{r,n}(t) = \frac{(R-t)}{R}B_{r,n-1}(t) + \frac{t}{R}B_{r-1,n-1}(t), \quad r=0,1,2,\dots,n \quad t\in[0,R], \tag{10}$$

The fractional derivative of the nth degree B-polynomials represent polynomials of degree $n-1$; they are denoted by

$$D^{\gamma}B_{r,n}(t) = \frac{n}{R}\left(B_{r-1,n-1}^{\gamma}(t) - B_{r,n-1}^{\gamma}(t)\right) \tag{11}$$

We will employ the Caputo derivative definition, which is a modification of the Riemann–Liouville definition. This definition can be advantageous in resolving certain initial value problems. The following multivariate fractional Taylor's theorem will be employed by us [25] in order to bound the absolute error.

Theorem 2.1. *For a compact and convex domain $D\subset\mathbb{R}$, let $D^{k\gamma}f\in C(D)$ for $k=0,1,\dots,m+1$ where*

$$D^{k\gamma}f = D^{k\gamma-n}D^nf, \quad n \text{ is the smallest integer exceeding } k\gamma$$
$$D^nf = \left(\Delta t\frac{\partial}{\partial t}\right)^n f.$$

If $t_0\in D$, then

$$f(x) = \sum_{k=0}^{m}\frac{D^{k\gamma}f(t_0)}{\Gamma(k\gamma+1)} + \frac{D^{(m+1)\gamma}f(\xi)}{\Gamma((m+1)\gamma+1)} = P_m^{\gamma}(t) + R_m^{\gamma}(\xi)$$

where $\xi = t_0+\theta\Delta t, 0<\theta<1$ and

$$P_m^{\gamma}(t) = \sum_{k=0}^{m}\frac{D^{k\gamma}f(t_0)}{\Gamma(k\gamma+1)} \text{ (Truncated mult. frac. Taylor series)} \tag{12}$$

$$R_m^{\gamma}(\xi) = \frac{D^{(m+1)\gamma}f(\xi)}{\Gamma((m+1)\gamma+1)} \text{ (Remainder term).}$$

3. Method of solutions

According to the definition of Bernstein polynomials in Section 2.2, the Bernstein series solution for Bernstein series solution of Eqs. (1) and (2) is obtained using the operational matrix of derivatives

$$u(t) = \mathbf{B}_n(t)\mathbf{A} \tag{13}$$

where

$$\mathbf{B}_n(t) = \begin{bmatrix} B_{0,n}(t) & B_{1,n}(t) & \cdots & B_{n,n}(t) \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} \tag{14}$$

$[\mathbf{B}_n(t)]^{\mathbf{T}}$ is expressed as follows:

$$[\mathbf{B}_n(t)]^{\mathbf{T}} = \begin{bmatrix} B_{0,n}(t) \\ B_{1,n}(t) \\ \vdots \\ B_{n,n}(t) \end{bmatrix} = \mathbf{X}(t)\mathbf{M}^{\mathbf{T}},$$

where

$$\mathbf{M} = \begin{pmatrix} m_{00} & m_{01} & \dots & m_{0n} \\ m_{10} & m_{11} & \dots & m_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n0} & m_{n1} & \dots & m_{nn} \end{pmatrix}, \quad \mathbf{X}(t) = [1 \quad t \quad t^2 \dots t^n], \tag{15}$$

$$m_{ij} = \begin{cases} \frac{(-1)^{j-i}}{R^j}\binom{n}{i}\binom{n-i}{j-i}, & i \le j \\ 0 \quad , & i > j \end{cases} \tag{16}$$

Meanwhile, we can write the approximate solution in (13) as follows:

$$u(t) = \mathbf{X}(t)\mathbf{M}^{\mathbf{T}}\mathbf{A}. \tag{17}$$

On the other hand, we can write the fractional derivative of (17) as follows:

$$D^{\gamma}u(t) = D^{\gamma}\mathbf{X}(t)\mathbf{M}^{\mathbf{T}}\mathbf{A}, \tag{18}$$

then

$$D^{\gamma}u(t) = D^{\gamma}\left[\mathbf{X}(t)\right]\mathbf{M}^{\mathbf{T}}\mathbf{A}, \tag{19}$$

Utilizing the Caputo definitions mentioned in (7), we can introduce the relationship between the matrix $\mathbf{X}(t)$ and its derivative $D^{\gamma}[\mathbf{X}(t)]$ as

$$D^{\gamma}[\mathbf{X}(t)] = \begin{bmatrix} 0 & \frac{\Gamma(2)}{\Gamma(2-\gamma)}t^{1-\gamma} & \frac{\Gamma(3)}{\Gamma(3-\gamma)}t^{2-\gamma} & \cdots & \frac{\Gamma(n+1)}{\Gamma(n+1-\gamma)}t^{n-\gamma} \end{bmatrix}. \tag{20}$$

the relationship in (20) can be written as

$$D^{\gamma}[\mathbf{X}(t)] = \mathbf{X}(t)\boldsymbol{\Psi}(t), \tag{21}$$

where

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2-\gamma)}t^{-\gamma} & 0 & \dots & 0 \\ 0 & 0 & \frac{\Gamma(3)}{\Gamma(3-\gamma)}t^{-\gamma} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{\Gamma(n+1)}{\Gamma(n+1-\gamma)}t^{-\gamma} \end{pmatrix}. \tag{22}$$

and

$$\mathbf{X}(t) = [1 \quad t \quad t^2 \dots t^n]. \tag{23}$$

Thus, the relationship in (19) can be written in the following manner:

$$u^{(\gamma)}(t) = \mathbf{X}(t)\boldsymbol{\Psi}(t)\mathbf{M}^{\mathbf{T}}\mathbf{A}, \tag{24}$$

Therefore, the part $u(\delta t-\omega)$ in the relation (1) may be expressed by converting it to matrix using the properties of Bernstein matrices and relationship (15) as,

$$\mathbf{X}(\delta t-\omega)=[1\quad (\delta t-\omega)\quad (\delta t-\omega)^2\dots(\delta t-\omega)^n]=\mathbf{X}(t)\Omega(\delta,\omega). \tag{25}$$

where

$$\Omega(\delta,\omega)=\begin{pmatrix} \binom{0}{0}(\delta)^0(\omega)^0 & \binom{1}{0}(\delta)^0(\omega)^1 & \binom{2}{0}(\delta)^0(\omega)^2 & \dots & \binom{n}{0}(\delta)^0(\omega)^n \\ 0 & \binom{1}{1}(\delta)^1(\omega)^0 & \binom{2}{1}(\delta)^1(\omega)^1 & \dots & \binom{n}{1}(\delta)^1(\omega)^{n-1} \\ 0 & 0 & \binom{2}{2}(\delta)^2(\omega)^0 & \dots & \binom{n}{2}(\delta)^2(\omega)^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \binom{n}{n}(\delta)^n(\omega)^0 \end{pmatrix}. \tag{26}$$

and for $\delta\neq 0$ and $\omega=0$

$$\Omega(\delta,0)=\begin{pmatrix} (\delta)^0 & 0 & 0 & \dots & 0 \\ 0 & (\delta)^1 & 0 & \dots & 0 \\ 0 & 0 & (\delta)^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & (\delta)^n \end{pmatrix}. \tag{27}$$

We can form the fractional derivative of relation (25) as

$$D^{\gamma}[\mathbf{X}(\delta t-\omega)]=\mathbf{X}(t)\Omega(\delta,\omega)\boldsymbol{\Psi}(t), \tag{28}$$

Therefore, the matrix relation of $u(\delta t-\omega)$ is obtained by employing the matrix forms in (17) and (25).

$$u(\delta t-\omega)=\mathbf{X}(\delta t-\omega)\mathbf{M}^{\mathbf{T}}\mathbf{A}. \tag{29}$$

We can express the fractional derivative of (29) as follows:

$$u^{(\gamma)}(\delta t-\omega)=D^{\gamma}[\mathbf{X}(\delta t-\omega)]\mathbf{M}^{\mathbf{T}}\mathbf{A}. \tag{30}$$

by utilizing relation (28) in (30), the following is derived:

$$u^{(\gamma)}(\delta t-\omega)=\mathbf{X}(t)\Omega(\delta,\omega)\boldsymbol{\Psi}(t)\mathbf{M}^{\mathbf{T}}\mathbf{A}, \tag{31}$$

We obtain the primary matrix equation by replacing the matrix relationships in (17), (24), and (31) into Eq. (1), as

$$\begin{cases} \mathbf{X}(t)\boldsymbol{\Psi}(t)\mathbf{M}^{\mathbf{T}}\mathbf{A}=f(t,\mathbf{X}(t)\mathbf{M}^{\mathbf{T}}\mathbf{A},\mathbf{X}(\delta t-\omega)\mathbf{M}^{\mathbf{T}}\mathbf{A}), 0\le t\le 1, \quad n-1<\gamma\le n \\ \mathbf{X}(\tau)\mathbf{M}^{\mathbf{T}}\mathbf{A}=\lambda_i \quad i=0,1,\dots,n-1, \quad 0\le\tau\le R. \end{cases} \tag{32}$$

We obtain the matrix $\mathbf{W}_{(\mathbf{n+1})\times(\mathbf{n+1})}$ by replacing the collocation points x_i in (32). These collocation points can be identified after implementing Chebyshev roots.

$$x_i=\frac{1}{2}+\frac{1}{2\cos((2i+1)\frac{\pi}{2n})}, \quad i=0,1,\dots,m-1. \tag{33}$$

Therefore, the equation in (32) becomes:

$$\mathbf{X}\boldsymbol{\Psi}\mathbf{M}^{\mathbf{T}}\mathbf{A}-\mathbf{X}\mathbf{M}^{\mathbf{T}}\mathbf{A}-\mathbf{X}(\delta,\omega)\mathbf{M}^{\mathbf{T}}\mathbf{A}=f(t), \tag{34}$$

we can write relation (34) as,

$$[\mathbf{X}\boldsymbol{\Psi}\mathbf{M}^{\mathbf{T}}-\mathbf{X}\mathbf{M}^{\mathbf{T}}-\mathbf{X}(\delta,\omega)\mathbf{M}^{\mathbf{T}}]\mathbf{A}=f(t), \tag{35}$$

it formulates (35) as the rudimentary matrix form yielded by

$$\mathbf{W}\mathbf{A}=\mathbf{F}, \tag{36}$$

where

$$\mathbf{W}=\mathbf{X}\boldsymbol{\Psi}\mathbf{M}^{\mathbf{T}}-\mathbf{X}\mathbf{M}^{\mathbf{T}}-\mathbf{X}(\delta,\omega)\mathbf{M}^{\mathbf{T}}. \tag{37}$$

We can write the initial conditions in (1) in the matrix forms in the following manner
$\mathbf{X}(\tau)\mathbf{M}^{\mathbf{T}}\mathbf{A}=\lambda_i \quad i=0,1,\dots,n-1, \quad 0\le\tau\le R.$

In the event the matrix $\mathbf{W}$ denotes an invertible square matrix, it becomes possible to identify the unknowns $\mathbf{A}=[a_0,a_1,\dots,a_n]$ by

$$\mathbf{A}=(\mathbf{W})^{-1}\mathbf{F} \tag{38}$$

4. Stability analysis and residual correction procedure

This section encompasses the estimation of stability concerning linear systems (1).

Let the system's solution as u^p,i.e., u^p signify the disturbing system's solution that follows:

$$\begin{cases} D^{(\gamma)}u(t) = f(t, u(t), u(\delta t - \omega)), 0 \le t \le 1, \quad n-1 < \gamma \le n \\ u^{(i)}(0) = \lambda_i \quad i = 0, 1, \dots, n-1. \end{cases} \tag{39}$$

The rudimentary matrix form is derived by

$$\mathbf{WA} = \mathbf{F}, \tag{40}$$

Thereafter, the implementation of this method yields the following:

$$\mathbf{WA} = \mathbf{F} + \Delta\mathbf{F}. \tag{41}$$

Let $\mathbf{A}^p$ = denote the perturbed solution of (41). Subsequently, the alteration in $\mathbf{A}$ can be bounded as: [26]

$$\frac{\|\Delta\mathbf{A}\|}{\|\mathbf{A}\|} \le cond(\mathbf{W})\frac{\|\Delta\mathbf{F}\|}{\|\mathbf{F}\|}.$$

As a second case, we consider the perturbed problem result of arithmetic operations

$$(\mathbf{W} + \Delta\mathbf{W})\,\mathbf{A} = \mathbf{F} + \Delta\mathbf{F}. \tag{42}$$

In the capacity of the same notation, the alteration in $\mathbf{A}$ triggered by perturbing the initials as well as arithmetic operations is bounded above in the following manner.

$$\frac{\|\Delta\mathbf{A}\|}{\|\mathbf{A}\|} \le \frac{cond(\mathbf{W})}{1 - cond(\mathbf{W})\frac{\|\Delta\mathbf{W}\|}{\|\mathbf{W}\|}}\left(\frac{\|\Delta\mathbf{W}\|}{\|\mathbf{W}\|} + \frac{\|\Delta\mathbf{F}\|}{\|\bar{\mathbf{F}}\|}\right).$$

Hence, BSSI can be bounded as follows for the case (41),

$$\begin{aligned} |u(t) - u^p(t)| &= \left|\mathbf{B}_n(t)\,(\mathbf{A} - (\mathbf{A} + \Delta\mathbf{A}))\right| \\ &\le \left\|\mathbf{B}_n(t)\right\|\,\|\Delta\mathbf{A}\| \\ &\le \left\|\mathbf{B}_n(t)\right\|\,cond(\mathbf{W})\frac{\|\Delta\mathbf{F}\|\,\|\mathbf{A}\|}{\|\mathbf{F}\|}. \end{aligned} \tag{43}$$

Consequently, we ascertain the manner in which minor alterations can impact the solution via the calculation of $cond(\mathbf{W})$. We can attain similar conclusions for (42).

With a view to constituting the error analysis by employing the residual correction procedure pertaining to this problem, let R_n be defined in the following manner:

$$R_n(x) := D^{(\gamma)}u(t) - f(t, u(t), u(\delta t - \omega)). \tag{44}$$

Subsequently, adding/subtracting the term R_n from Eq. (44) yields the following problem for the absolute error

$$e_n^{\alpha}(x) = f(t, e_n(t), e_n(\delta t - \omega)). \tag{45}$$

where $e_n = u - u_n$ associated with this initial condition

$$e_n^{(\alpha)}(\delta) = 0. \tag{46}$$

We get an approximate solution using the approach in Eq. (45) with the condition (46), which is signified by $e_{n,m}$, for the absolute error, where m represents the degree of approximation.

Notably, $u_{n,m} := u_n + e_{n,m}$ denotes another approximate solution, referred to as the corrected solution, and its error function is $e_{n,m}$. In case $\|e_n - e_{n,m}\| < \|u - e_n\|$, $u_{n,m}$ can be deemed an improved approximation than u_n in the norm. Meanwhile, we can estimate e_n by $e_{n,m}$ whenever $\|e_{n,n} - e_{n,n}^{m,m}\| < \epsilon$ is small.

5. Numerical results and discussion

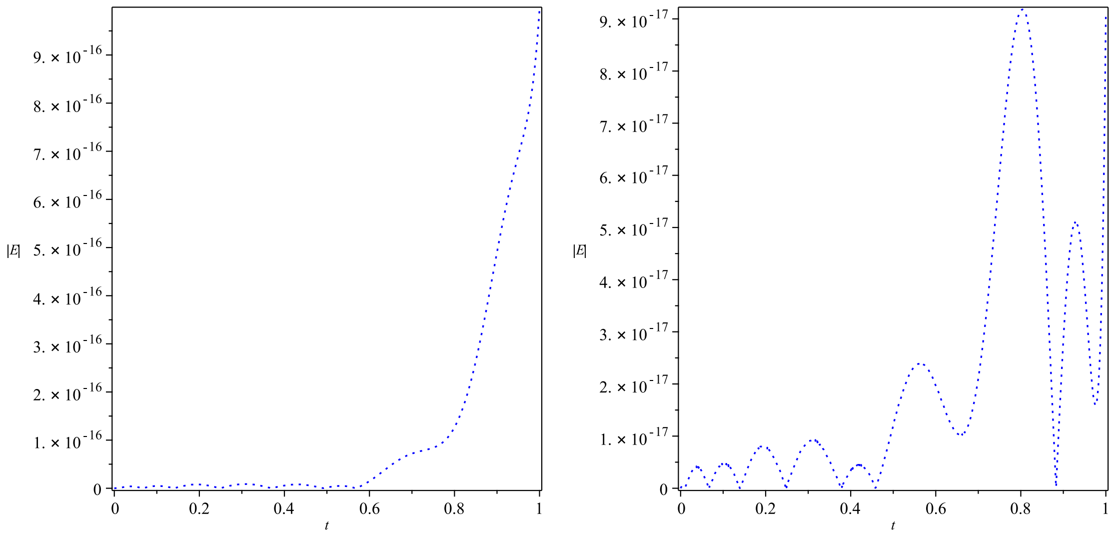
This section provides four examples to highlight the strengths and effectiveness of our method. The first two examples will be solved as follows: To begin with, we will solve as Problem 1; subsequently, we will solve two examples as Problem 2.

Example 1. Let us take into consideration the fractional delay differential equation [12]

$$u^{(0.9)}(t) = \frac{2t^{1.1}}{\Gamma(2.1)} - \frac{t^{0.1}}{\Gamma(1.1)} + u(t - 0.1) - u(t) + 0.2t - 0.11 \tag{47}$$

We can denote the initial condition as follows:

$$u(t) = 0, \quad t \le 0.$$



Absolute error of our solutions for n = 12, and ω=0.2. Absolute error of our solutions for n = 12, and ω=0.5.

Fig. 1. The absolute error for Example 2, with $n = 12$, $t = 1$ and $\omega = 0.2, 0.5$.

This problem's exact solution is

$$u(t) = t^2 - t. \tag{48}$$

After implementing the technique in Section 3, with n = 2, we obtain the fundamental matrices for Eq. (47) in the following manner:

$$\mathbf{M^T} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{pmatrix}, \quad \mathbf{X}(t) = [1 \quad t \quad t^2],$$

$$\Omega(\delta, \omega) = \begin{pmatrix} 1 & -0.1 & 0.01 \\ 0 & 1 & -0.2 \\ 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{\Psi}(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1.05t^{-0.9} & 0 \\ 0 & 0 & 1.91t^{-0.9} \end{pmatrix}$$

We identify the values of unknowns a_i in (14) as:

$$\mathbf{A} = \begin{bmatrix} a_0 \\ a_1 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.5 \\ 0 \end{bmatrix}$$

We get the approximate solution as follows by substituting these unknowns a_i into Eq. (13),

$$u(t) = t^2 - t.$$

which, in turn, denotes the exact solution, in the Eq. (48).

Our method helps derive the exact solution. On the other hand, we found that the methods used to solve the problem (47), namely, FAM, three-term NPCM, and the new method (L1-PCM), suffered from errors; refer to [12]. This exemplifies the effectiveness and accuracy of the existing method.

Example 2. Let us consider the fractional delay differential equation [27]

$$u^{(\gamma)}(t) = -u(t) + \frac{\omega}{2}u(\omega t) - \frac{\omega}{2}e^{-\omega t}, \quad 0 \le t \le 1 \quad 0 \le \gamma \le 1 \tag{49}$$

The initial condition is as follows:

$$u(0) = 1.$$

This problem's exact solution when $\gamma = 1$ is $u(t) = e^{-t}$.

The fundamental matrices equations for Eq. (49) are obtained by applying the technique in Section 3, with $n = 12$; in addition, we obtain and substitute a_i in Eq. (13) in order to identify the approximate solution for this problem.

The values of unknowns a_i in (14) are found as follows:

$$\mathbf{A} = \begin{bmatrix} 1 & 0.91\overline{60} & 0.84\overline{90} & 0.77\overline{96} & \cdots & 0.36\overline{78} \end{bmatrix}. \tag{50}$$

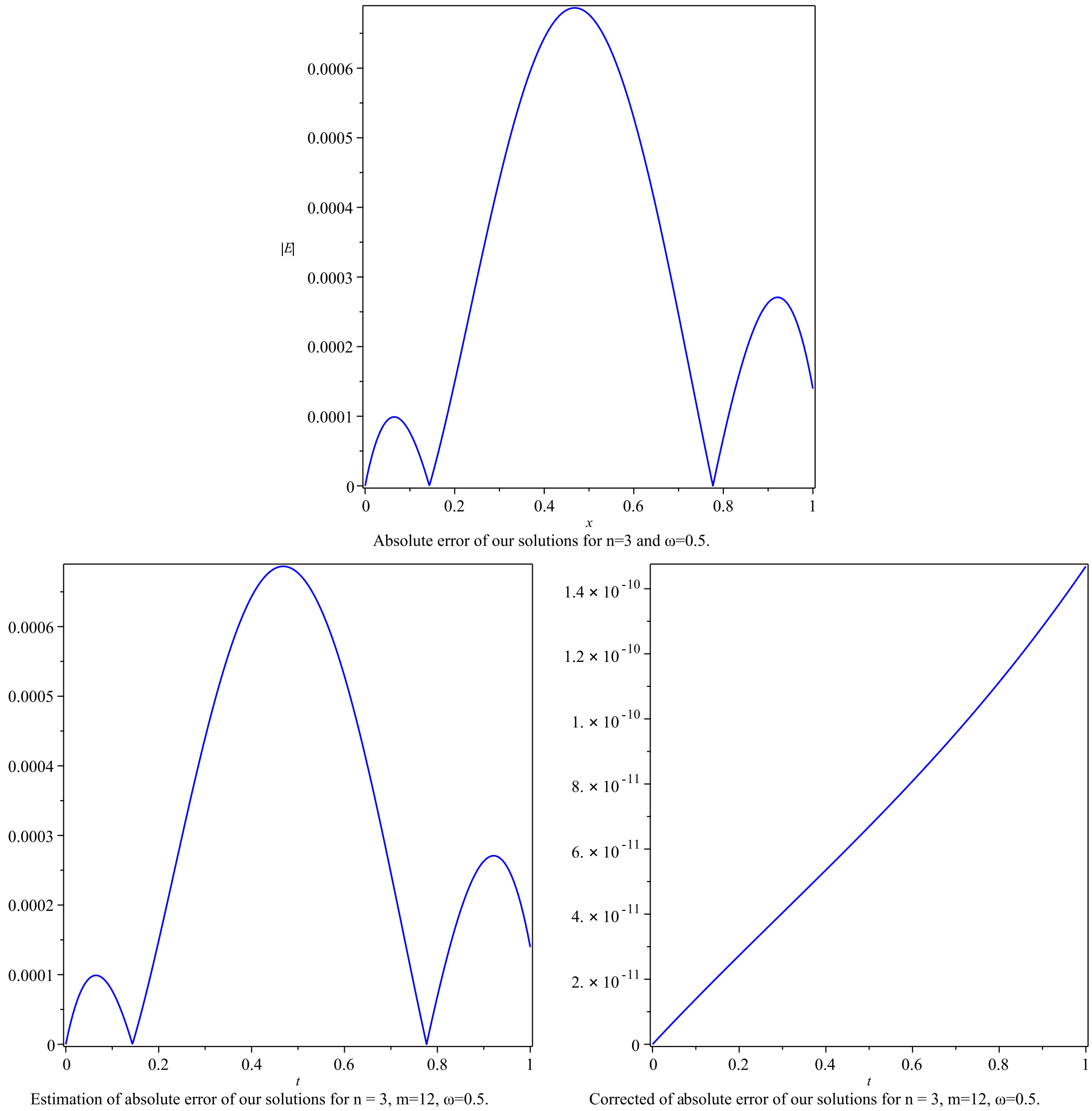
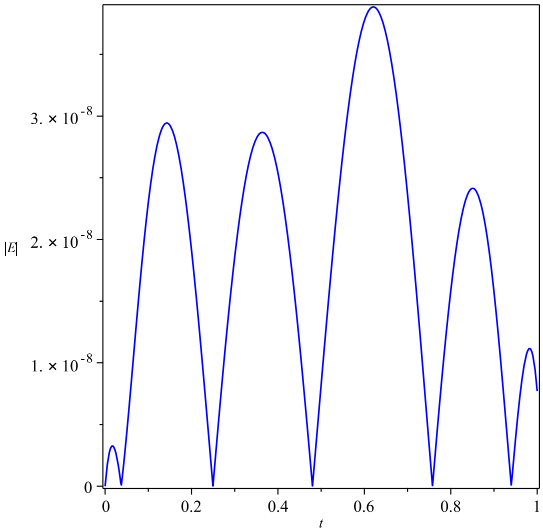


Fig. 2. The error correction procedure for Example 2, with $n = 3$ and $m = 12$ at time $t = 1$ and $\omega = 0.5$.

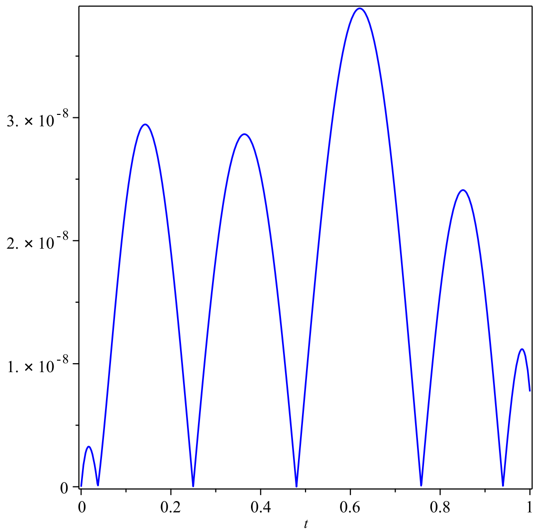
Table 1
Comparison of the absolute error utilizing the method outlined in Section 3 and those produced by means of the Bernoulli wavelets method [27] and collocation methods [28] at $\gamma = 1$ for Example 2.

t	Present method		Other methods	
	$n = 6$	$n = 12$	[27]	[28]
1/4	1.56×10^{-10}	5.50×10^{-19}	1.05×10^{-8}	1.08×10^{-5}
1/8	2.83×10^{-8}	2.89×10^{-18}	5.79×10^{-9}	3.81×10^{-5}
1/16	9.18×10^{-9}	8.70×10^{-19}	2.00×10^{-8}	1.26×10^{-5}
1/32	1.53×10^{-9}	3.82×10^{-18}	3.70×10^{-9}	4.09×10^{-5}
1/64	3.24×10^{-9}	1.17×10^{-18}	2.30×10^{-8}	1.20×10^{-5}

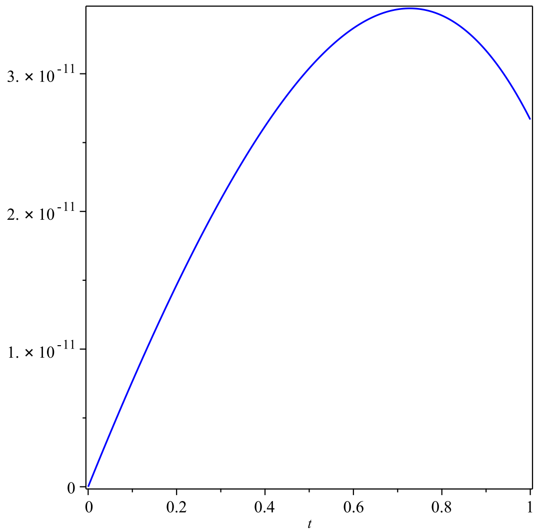
Table 1 presents the approximate solution's values and compares them with other methods at various stages. Furthermore, Figs. 1, 2, and 3, illustrate the plots of the exact solution, the approximate solution, as well as the corrected approximate solution, respectively. Table 2 displays the stability results premised on the method employed by us.



Absolute error of our solutions for n = 6, ω=0.5.



Estimation of absolute error of our solution for n=6, m=12, ω=0.5.



Corrected of absolute error of our solutions for n = 6, m=12, ω=0.5.

Fig. 3. The error correction procedure for Example 2, with $n = 6$ and $m = 12$ at time $t = 1$ and $\omega = 0.5$.

Table 2
Stability results of the system derived through the existing method for Example 2.

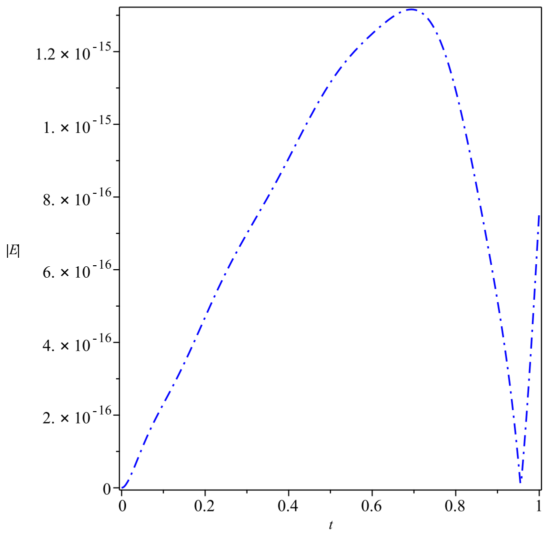
	$n = 2$	$n = 4$	$n = 8$
$cond(W)$	4.72	21	133.77
$\|\Delta A\|$	3.02×10^{-1}	9.80×10^{-1}	9.99×10^{-1}
$\|A\|$	0.47	0.75	0.87
$\|\Delta G\|$	10^{-16}	10^{-16}	10^{-16}
$\|G\|$	9.7×10^{-2}	9.95×10^{-2}	9.98×10^{-2}
Upper bound obtained by (43)	1.12×10^{-15}	1.92×10^{-14}	1.96×10^{-12}
$\|u_n - u_n^p\|$	1.0×10^{-16}	2.0×10^{-16}	3.0×10^{-16}

Example 3. Let us take into consideration the fractional delay differential equation [16]

$$u^{(\gamma)}(t) = 1 - 2u^2(\frac{t}{2}) \quad 0 \leq t \leq 1 \quad 1 \leq \gamma \leq 2 \tag{51}$$

The initial condition is as follows:

$$u(0) = 1, \quad u'(0) = 0$$



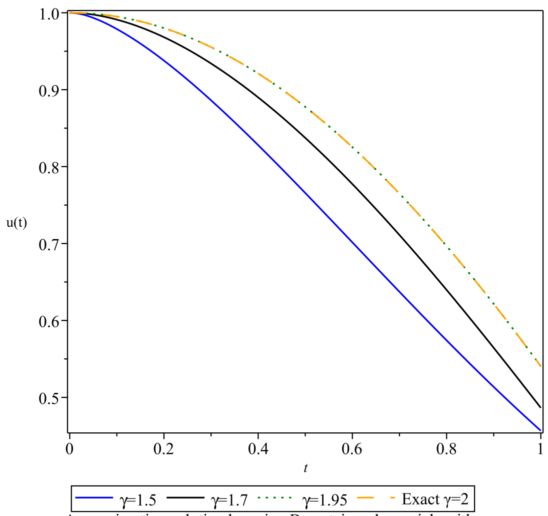


Fig. 4. The absolute error for Example 3, with $n = 12$, $t = 1$ and $\gamma = 2$, and approximate solutions for varying γ values with exact.

Table 3
Comparison of the absolute error utilizing the method outlined in Section 3 along with those produced by the Chebyshev Spectral method [16] and MLWM methods [29] and OMBWM [27] at $\gamma = 2$ for Example 3.

t	Chebyshev spectral [16]	MLWM [29] $m = 20$	OMBWM [27] $\alpha = 2$	Present method $n = 12$
0.0	1.11×10^{-16}	2.11×10^{-8}	1.62×10^{-11}	0
0.2	1.98×10^{-12}	2.09×10^{-8}	3.30×10^{-13}	4.69×10^{-17}
0.4	2.13×10^{-12}	2.08×10^{-8}	4.17×10^{-9}	9.06×10^{-16}
0.6	2.82×10^{-12}	2.04×10^{-8}	1.08×10^{-10}	1.24×10^{-15}
0.8	3.63×10^{-12}	2.00×10^{-8}	1.62×10^{-9}	1.09×10^{-15}

This problem's exact solution when $\gamma = 2$ is $u(t) = \cos t$.

The results are tabulated in Table 3 by applying the presented method for $n = 12$; it becomes apparent that the proposed method proves to be more precise in comparison to other methods, in [16,27,29]. Fig. 4 shows absolute errors with $n = 12, \gamma = 2$ as well as the approximate solutions for varying values of γ with exact solutions.

Example 4. Let us examine the system of fractional delay differential equation [16]

$$u_1^{(\gamma)}(t) = u_1(t) - u_2(t) + u_1(0.5t) - e^{0.5t^\gamma} + e^{-t^\gamma}$$
$$u_2^{(\gamma)}(t) = -u_1(t) - u_2(t) - u_2(0.5t) + e^{-0.5t^\gamma} + e^{t^\gamma} \tag{52}$$

The initial conditions are $u_1(0) = 1, \quad u_2(0) = 1$.

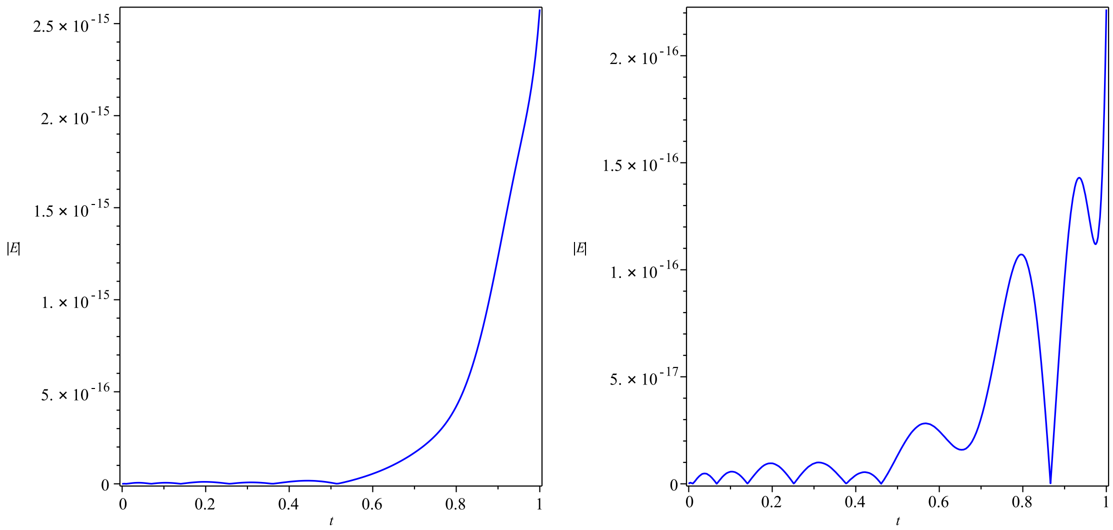
This problem's exact solution when $\gamma = 1$ is $u_1(t) = e^t$ and $u_2(t) = e^{-t}$.

The fundamental matrix equation for Eq. (52) is obtained by applying the technique in Section 3, and the findings can be seen in Tables 5 and 6. Absolute error analysis with the findings of residual correction procedure are visually represented in Figs. 5–7. Table 4 illustrates the stability results for $u_1(t)$ and $u_2(t)$ on the basis of the method used by us.

The problem is approached by [16] applying the Chebyshev spectral methods and [15] that implemented the tau spectral method (TSM) for its solution. In Tables 5 and 6, the absolute errors of $u_1(t)$ and $u_2(t)$ are compared with $n = 12$ against the best results attained by using the RPSM [30], TSM [15,16].

6. Conclusions

This study demonstrates the efficacious utilization of operational matrices of differentiation approach by means of Bernstein series in order to solve a class of FDDEs. The comparison between our proposed method and others is presented in Section 5, where it is evident that our algorithm yields more accurate results compared to methods such as the Bernoulli wavelets method, the Laguerre wavelets method, collocation methods, and the Chebyshev spectral method. The numerical experiments demonstrate a strong agreement between theoretical predictions and numerical outcomes, as evidenced by Tables 1–6 and Figs. 1–7. Throughout



Absolute error of $u_1(t)$, with n=12, and γ=1. Absolute error of approximate solution $u_2(t)$, with n=12, and γ=1.

Fig. 5. The absolute error for Example 4, with $n = 12$, $\gamma = 1$.

Table 4
Stability findings derived via our method of the pantograph system of equations shown in Example 4.

	u_i	$n = 2$	$n = 4$	$n = 8$
$cond(W)$	u_1	41.14	243.61	7766.8
	u_2	3.14	41.23	83.5.77
$\|\Delta A\|$	u_1	1.66	2.51	1.79
	u_2	3.48	5.02	9.92
$\|A\|$	u_1	2.78	2.71	2.37
	u_2	0.34	0.80	2.71
$\|\Delta G\|$	u_2	10^{-16}	10^{-16}	10^{-16}
	u_2	10^{-16}	10^{-16}	10^{-16}
$\|G\|$	u_1	1.10	1.25	1.26
	u_2	3.00	3.26	3.30
Upper bound obtained by (43)	u_2	1.66×10^{-14}	9.75×10^{-13}	5.11×10^{-10}
	u_2	1.31×10^{-15}	1.88×10^{-13}	6.12×10^{-12}
$\|u_n - u_n^p\|$	u_1	1.0×10^{-16}	2.0×10^{-16}	3.0×10^{-16}
	u_2	1.0×10^{-16}	2.0×10^{-16}	3.0×10^{-16}

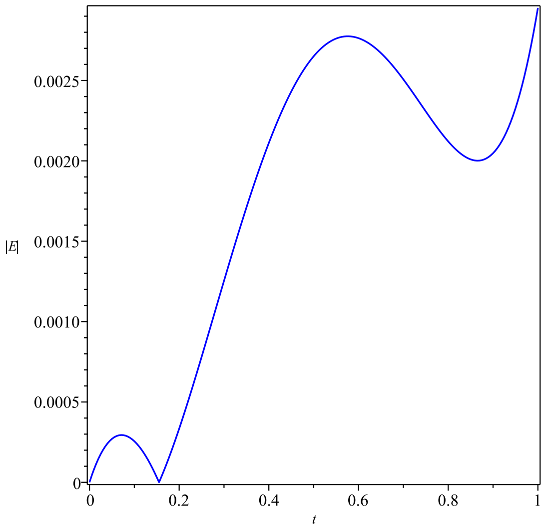
Table 5
Comparison of the absolute error for $u_1(t)$ utilizing the method outlined in Section 3 along with those produced by the RPSM method [30], TSM methods [15] and Chebyshev spectral [16] at $\gamma = 1$ for Example 4.

$u_1(t)$				
t	RPSM [30]	TSM [15]	Chebyshev spectral [16]	Present method, $n = 12$
0.2	2.60×10^{-9}	4.50×10^{-8}	5.38×10^{-11}	1.07×10^{-17}
0.4	3.42×10^{-7}	5.75×10^{-8}	1.63×10^{-9}	1.11×10^{-17}
0.6	6.00×10^{-6}	6.02×10^{-8}	1.96×10^{-9}	5.44×10^{-17}
0.8	4.61×10^{-5}	4.62×10^{-8}	8.02×10^{-10}	4.20×10^{-16}
1	2.76×10^{-4}	1.27×10^{-8}	2.37×10^{-9}	2.57×10^{-15}

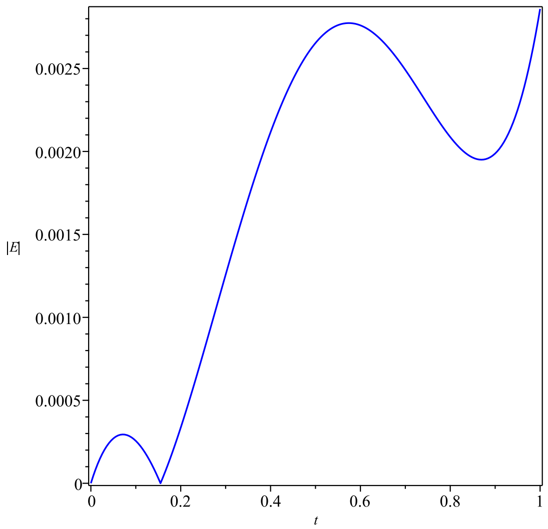
the illustrations presented so far, we have demonstrated the fundamental matrices of integration and differentiation $\mathbf{\Psi}, \mathbf{\Omega}(\delta, \omega), \mathbf{W}$ and $\mathbf{A}$ generated on the basis of our techniques. Furthermore, this novel approach can also be leveraged for the option pricing model solution.

Declaration of competing interest

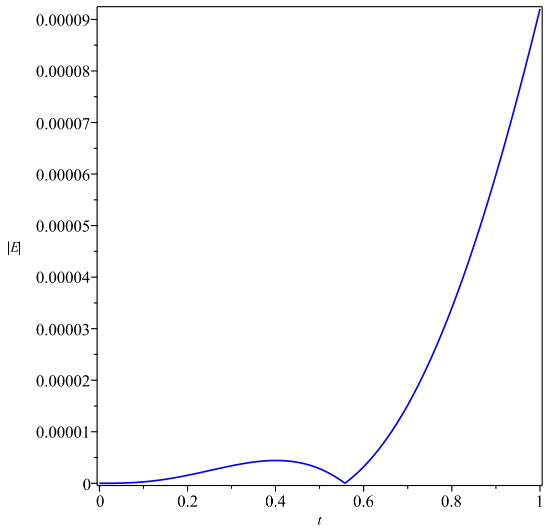
The author declare that they have no known competing financial interest or personal relationship that could have appeared to influence the work reported in this paper.



Absolute error of $u_1(t)$, with n=3, and γ=1.



Estimation of the absolute error of $u_1(t)$, with n=3, m=12, and γ=1.



Corrected absolute error of $u_1(t)$, with n=3, m=12, and γ=1.

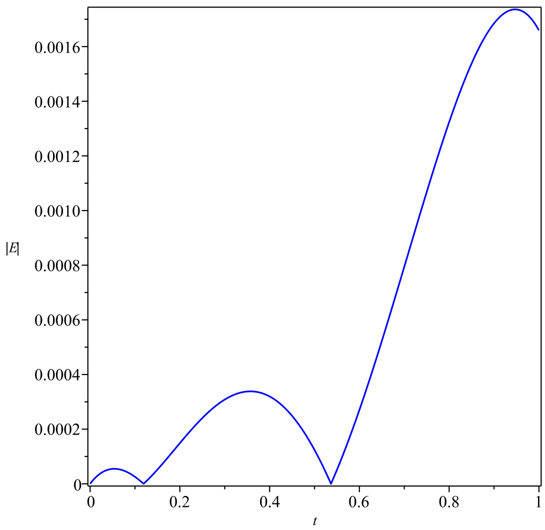
Fig. 6. The error correction procedure for Example 4 $u_1(t)$ with $n = 3$ and $m = 12$.

Table 6
Comparison of the absolute error for $u_2(t)$ utilizing the method outlined in Section 3 along with those produced by the RPSM method [30], TSM methods [15] and Chebyshev spectral [16] at $\gamma = 1$ for Example 4.

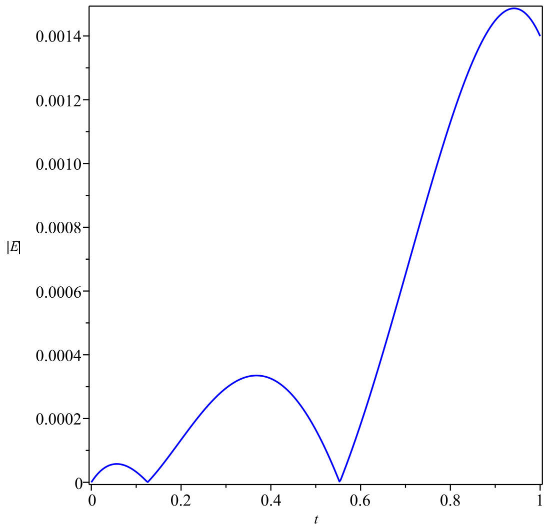
$u_2(t)$				
t	RPSM [30]	TSM [15]	Chebyshev spectral [16]	Present method, $n = 12$
0.2	2.47×10^{-9}	2.14×10^{-8}	4.63×10^{-10}	9.55×10^{-18}
0.4	2.60×10^{-8}	2.60×10^{-8}	2.43×10^{-10}	4.02×10^{-18}
0.6	1.63×10^{-8}	1.63×10^{-8}	3.91×10^{-10}	2.45×10^{-17}
0.8	1.12×10^{-8}	1.12×10^{-8}	5.07×10^{-10}	1.60×10^{-16}
1	2.97×10^{-9}	2.97×10^{-9}	1.05×10^{-9}	2.21×10^{-16}

Data availability

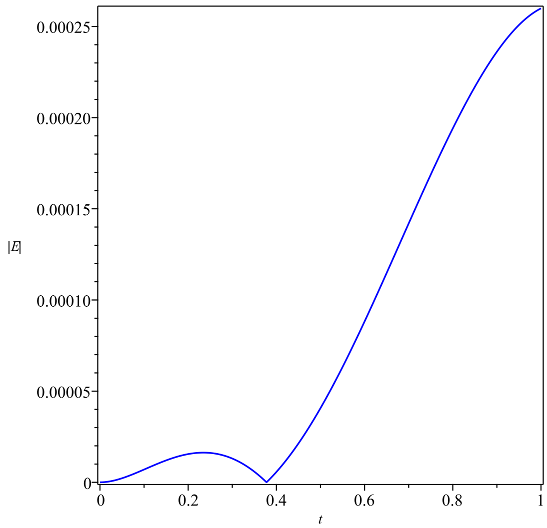
No data was used for the research described in the article.



Absolute error of $u_2(t)$, with n=3, and γ=1.



Estimation of absolute error of $u_2(t)$, with n=3, m=12, and γ=1.



Corrected absolute error of $u_2(t)$, with n=3, m=12, and γ=1.

Fig. 7. The error correction procedure for Example 4 $u_2(t)$ with $n = 3$ and $m = 12$.

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