1-1-2024

Non-Carathéodory analytic functions with respect to symmetric points

Daniel Breaz
Universitatea 1 Decembrie 1918 din Alba Iulia

Kadhavoor R. Karthikeyan
National University of Science & Technology, Oman

Elangho Umadevi
Zayed University

Taylor & Francis
Taylor & Francis Group

Mathematical and Computer Modelling of Dynamical Systems

Methods, Tools and Applications in Engineering and Related Sciences

ISSN: (Print) (Online) Journal homepage: www.tandfonline.com/journals/nmcm20

Non-Carathéodory analytic functions with respect to symmetric points

Daniel Breaz, Kadhavoor R. Karthikeyan & Elangho Umadevi

To cite this article: Daniel Breaz, Kadhavoor R. Karthikeyan & Elangho Umadevi (2024) Non-Carathéodory analytic functions with respect to symmetric points, Mathematical and Computer Modelling of Dynamical Systems, 30:1, 266-283, DOI: 10.1080/13873954.2024.2341691

To link to this article: https://doi.org/10.1080/13873954.2024.2341691

Published online: 21 Apr 2024.

MATHEMATICAL AND COMPUTER MODELLING OF DYNAMICAL SYSTEMS
2024, VOL. 30, NO. 1, 266–283
https://doi.org/10.1080/13873954.2024.2341691

Non-Carathéodory analytic functions with respect to symmetric points

Daniel Breaz[a], Kadhavoor R. Karthikeyan[b] and Elangho Umadevi[c]

[a]Department of Mathematics, "1 Decembrie 1918" University of Alba Iulia, Alba Iulia, Romania; [b]Department of Applied Mathematics and Science, National University of Science & Technology, Muscat, Oman; [c]Department of Mathematics and Statistics, College of Natural and Health Sciences, Zayed University, Abu Dhabi, UAE

ABSTRACT
The authors introduce new classes of analytic function with respect (η, τ)-symmetric points subordinate to a domain that is not Carathéodory. To use the existing infrastructure or framework, usually, the study of analytic function have been limited to a differential characterization subordinate to functions which are Carathéodory. Here, we try to obtain various interesting properties of functions which are not Carathéodory. Integral representation, interesting conditions for starlikeness and inclusion relations for functions in these classes are obtained.

ARTICLE HISTORY
Received 13 December 2023
Accepted 29 March 2024

KEYWORDS
Univalent function; symmetrical functions; differential subordination

1. Introduction, definitions and preliminaries

Let $\mathcal{A}$ be the class of function of the form

$$\varphi(\omega) = \omega + \sum_{n=2}^{\infty} a_n \omega^n, \tag{1.1}$$

which are analytic in the unit disc $\mathbb{U} = \{\omega : |\omega| < 1\}$. Let $\mathcal{S}$ denote the class of functions $\varphi \in \mathcal{A}$ which are univalent in $\mathbb{U}$. We call $\mathcal{P}$ to denote the class of functions with normalization $p(0) = 1$ which satisfies $\mathrm{Re}(p(\omega)) > 0$, $\omega \in \mathbb{U}$. Starlike and convex functions, the well-known geometrically defined subclasses of $\mathcal{S}$ have the following analytic characterizations respectively

$$\frac{\omega\varphi'(\omega)}{\varphi(\omega)} \in \mathcal{P} \quad \text{and} \quad 1 + \frac{\omega\varphi''(\omega)}{\varphi'(\omega)} \in \mathcal{P}.$$

We denote the class of starlike and convex functions by $\mathcal{S}^*$ and $\mathcal{C}$ respectively. Ma-Minda (Ma and Minda 1992) studied an analytic function ψ which satisfies the conditions

(i) $\mathrm{Re}\ \psi > 0,\ \mathbb{U}$;
(ii) $\psi(0) = 1,\ \psi'(0) > 0$;

CONTACT Kadhavoor R. Karthikeyan karthikeyan@nu.edu.om Department of Applied Mathematics and Science, College Of Engineering (Al Hail Caledonian Campus), National University of Science & Technology, P.O Box 2322, CPO Seeb 111 Al Hail, Muscat, Oman

(iii) (iii) ψ maps $\mathbb{U}$ onto a starlike region with respect to 1 and symmetric with respect to the real axis.

Also, they assumed that $\psi(z)$ has a series expansion of the form

$$\psi(\omega) = 1 + L_1\omega + L_2\omega^2 + L_3\omega^3 + \cdots, \qquad (L_1 > 0;\ \omega \in \mathbb{U}). \tag{1.2}$$

and introduced and studied the following subclasses of using subordination of analytic functions:

$$\mathcal{S}^*(\psi) := \left\{\varphi \in \mathcal{A} : \frac{\omega\varphi'(\omega)}{\varphi(\omega)} \prec \psi(\omega)\right\}$$

and

$$\mathcal{C}(\psi) := \left\{\varphi \in \mathcal{A} : \left(1 + \frac{\omega\varphi''(\omega)}{\varphi'(\omega)}\right) \prec \psi(\omega)\right\}.$$

By choosing ψ to map unit disc on to some specific regions like parabolas, cardioid, lemniscate of Bernoulli, booth lemniscate in the right-half of the complex plane, various interesting subclasses of starlike and convex functions can be obtained. For detailed study, refer to (Gandhi 2020; Dziok et al. 2011b, 2011a, 2013; Mendiratta et al. 2014; Raina and Sokół 2015, 2016, 2019; Srivastava et al. 2019, 2019, 2019, 2019; Mustafa and Murugusundaramoorthy 2021; Khan et al. 2022; Mustafa and Korkmaz 2022; Araci et al. 2023).

From the above discussion, it can be seen that the entire architecture supports the study of analytic functions that are subordinate to Carathéodory function. Here in this study, we will deviate by introducing certain differential characterization subordinate to a *non-Carathéodory functions*.

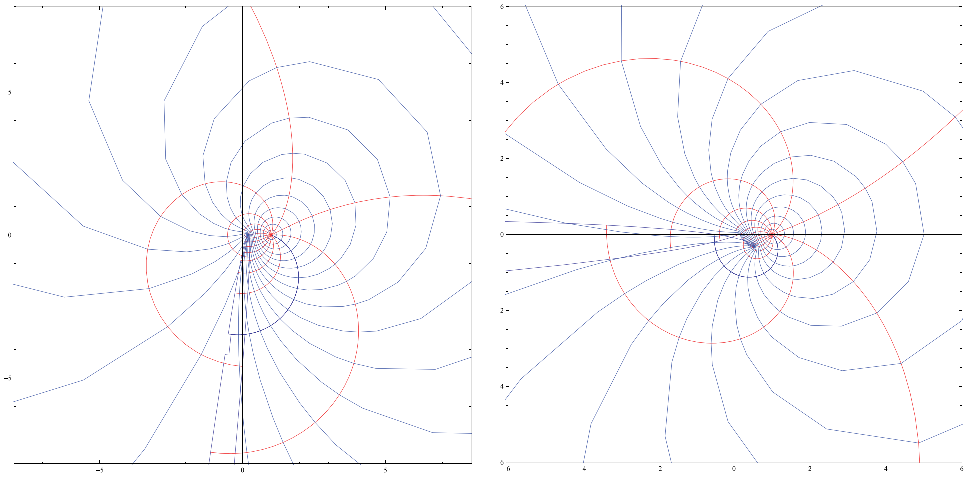
To begin with, we let $\mathcal{NP}$ to denote the class of functions that are analytic in the unit disc and equals 1 at $\omega = 0$. Both Carathéodory and non-Carathéodory functions satisfy the same normalization $p(0) = 1$, the only difference is that the requirement of the function to map the unit disc onto a right-half plane is relaxed in case of non-Carathéodory functions. Recently, Karthikeyan et al. (2023) introduced a class belonging to a class of *non-Carathéodory functions* $\mathcal{NP}$ defined by

$$\Lambda[\alpha, \beta; p(\omega)] = \frac{e^{-i\alpha}[\cos\alpha + i\sin\alpha\, p(\omega)]}{1 - 2\cos\beta\omega e^{-i\alpha} + e^{-2i\alpha}\omega^2}, \tag{1.3}$$

with $\alpha, \beta \in [0, \pi]$ and $p(\omega) \in \mathcal{P}$. Here, in this paper, we slightly modify the equation (1.3) to accommodate or unify the studies of well-known classes of analytic functions, which we define as follows.

$$\Lambda^*[\alpha, \beta; p(\omega)] = \frac{e^{-i\alpha}[\cos\alpha + i\sin\alpha(1 - \omega^2)p(\omega)]}{1 - 2\cos\beta\omega e^{-i\alpha} + e^{-2i\alpha}\omega^2}. \tag{1.4}$$

It can be easily seen that for a choice of $\alpha = \beta = \frac{\pi}{2}$, $\Lambda^*[\alpha, \beta; p(\omega)] = p(\omega) \in \mathcal{P}$. Figure 1a,b illustrates the impact of $\Lambda^*[\frac{\pi}{3}, 0; p(\omega)]$ on the regions $p(\omega) = \frac{1+\omega}{1-\omega}$ and $p(\omega) = \sqrt{1+\omega}$ respectively. Further, the images shows that function $\Lambda^*[\alpha, \beta; p(\omega)]$ belongs to the class which is not Carathéodory.



(a) $p(\omega) = \frac{1+\omega}{1-\omega}$ (b) $p(\omega) = \sqrt{1+\omega}$

Figure 1. Mapping of $|\omega| < 1$ under $\Lambda^*[\pi/3, 0; p(\omega)]$.

Mittag-Leffler function, a special transcendental function has been on the spotlight due to its role in treating problems related to integral and differential equations of fractional order. Refer to Srivastava (2021c, 2021a, 2021b), Srivastava (1968), Srivastava and Tomovski (2009) and Srivastava et al. (2018, 2019, 2022) for detailed studies which involved Mittag-Leffler function. The function $E^{\rho}_{\theta,\vartheta}(\omega)$ is popularly known as Prabhakar function or generalized Mittag-Leffler three parameter function. Explicitly, the generalized Mittag-Leffler three parameter function is defined by

$$E^{\rho}_{\theta,\vartheta}(\omega) = \sum_{n=0}^{\infty} \frac{(\rho)_n \omega^n}{\Gamma(\theta n + \vartheta) n!}, \quad (\omega, \theta, \vartheta, \rho \in \mathbb{C}, Re(\theta) > 0). \tag{1.5}$$

where $\mathbb{C}$ denotes the sets of complex numbers and $(x)_n$ will be used to denote the usual Pochhammer symbol.

Using the Mittag-leffler function, Murat et al. (2023) defined the following operator $D^m_r(\theta, \vartheta, \rho)\varphi : \mathbb{U} \to \mathbb{U}$ by

$$D^m_r(\theta, \vartheta, \rho)\varphi(\omega) = \omega + \sum_{n=2}^{\infty} \left[n + \frac{r}{2}(1 + (-1)^{n+1})\right]^m \frac{\Gamma(\vartheta)(\rho)_{n-1}}{\Gamma(\vartheta + \theta(n-1))(n-1)!} a_n \omega^n, \tag{1.6}$$

$$(m, r \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}, \omega, \theta, \vartheta, \rho \in \mathbb{C}, Re(\theta) > 0).$$

The operator $D^m_r(\theta, \vartheta, \rho)\varphi$ was motivated by the operator $D^m_r\varphi(\omega)$ defined by Ibrahim and Darus (2019) and Ibrahim (2020).

It is well known that if $\varphi(\omega)$ given by (1.1) is in $\mathcal{S}$, then $[\varphi(\omega^\tau)]^{1/\tau}$, ($\tau$ is a positive integer) is also in $\mathcal{S}$. For every integer η, a function $\varphi \in A$ is said to be (η, τ)-symmetrical if for each $\omega \in \mathbb{U}$

$$\varphi(\varepsilon\omega) = \varepsilon^{\eta}\varphi(\omega), \tag{1.7}$$

where $\tau \geq 2$ is a fixed integer, $\eta = 0, 1, 2, \ldots, \tau - 1$ and $\varepsilon = \exp(2\pi i/\tau)$. The family of (η, τ)-symmetrical functions denoted by $\mathcal{F}_{\tau}^{\eta}$ was defined and studied by Liczberski and Po ubin′ski in Liczberski and Połubiński (1995). We observe that $\mathcal{F}_{2}^{1}, \mathcal{F}_{2}^{0}$ and $\mathcal{F}_{\tau}^{1}$ are well-known families of odd functions, even functions and τ-symmetrical functions respectively.

Also, for every integer η, let $\varphi_{\eta,\tau}(\omega)$ be defined by the following equality

$$\varphi_{\eta,\tau}(\omega) = \frac{1}{\tau}\sum_{\nu=0}^{\tau-1}\frac{\varphi(\varepsilon^{\nu}\omega)}{\varepsilon^{\nu\eta}}, \quad (\varphi \in \mathcal{A}). \tag{1.8}$$

Obviously, $\varphi_{\eta,\tau}(\omega)$ inherits the all linearity properties $\varphi(\omega)$. The characterization (1.8) was first presented by Liczberski and Po ubin′ski in (Liczberski and Połubiński 1995).

The following identities can be derived from (1.8), provided ν is an integer,:

$$\begin{aligned} \varphi_{\eta,\tau}(\varepsilon^{\nu}\omega) &= \varepsilon^{\nu\eta}\varphi_{\eta,\tau}(\omega), \\ \varphi'_{\eta,\tau}(\varepsilon^{\nu}\omega) &= \varepsilon^{\nu\eta-\nu}\varphi'_{\eta,\tau}(\omega) = \tfrac{1}{\tau}\sum_{\nu=0}^{\tau-1}\tfrac{\varphi'(\varepsilon^{\nu}\omega)}{\varepsilon^{\nu\eta-\nu}}, \\ \varphi''_{\eta,\tau}(\varepsilon^{\nu}\omega) &= \varepsilon^{\nu\eta-2\nu}\varphi''_{\eta,\tau}(\omega) = \tfrac{1}{\tau}\sum_{\nu=0}^{\tau-1}\tfrac{\varphi''(\varepsilon^{\nu}\omega)}{\varepsilon^{\nu\eta-2\nu}}. \end{aligned} \tag{1.9}$$

We assume that $\tau \in \mathbb{N}$, $\varepsilon = \exp(2\pi i/\tau)$ and

$$D_r^m(\theta, \vartheta, \rho)\varphi_{\eta,\tau}(\omega) = \frac{1}{\tau}\sum_{\nu=0}^{\tau-1}\varepsilon^{-\nu\eta}\left[D_r^m(\theta, \vartheta, \rho)\varphi(\omega)\right] = w + \cdots, \tag{1.10}$$

From (1.10), we can get

$$D_r^m(\theta, \vartheta, \rho)\varphi_{\eta,\tau}(\omega) = \sum_{\ell=1}^{\infty} a_{\ell}\Phi_{\ell}\Gamma_{\ell,\eta}w^{\ell}, \quad (a_1 = \Phi_1 = 1), \quad \Gamma_{\ell,\eta} = \frac{1}{\tau}\sum_{\nu=0}^{\tau-1}\varepsilon^{(\ell-\eta)\nu}, \tag{1.11}$$

where

$$\Phi_n = \left[n + \frac{r}{2}\left(1 + (-1)^{n+1}\right)\right]^m \frac{\Gamma(\vartheta)(\rho)_{n-1}}{\Gamma(\vartheta + \theta(n-1))(n-1)!}, n \geq 2$$

We will define a comprehensive subclass of analytic functions involving the well-known Mittag-Leffler function. The major deviation of this paper is that, we have obtained coefficient inequalities, inclusion relationships, integral representation and closure property using differential subordination for a subclass of *non-Carathéodory functions*.

Motivated by (Srivastava et al. 2018; Karthikeyan et al. 2021), we now define the following:

Definition 1.1. For $0 \leq \delta \leq 1$, $\theta, \vartheta \in \mathbb{C}$, $Re(\theta) > 0$ and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\varphi \in \mathcal{A}$ is said to be in the class $\mathcal{C}^m_{(\eta,\tau)}(\theta, \vartheta, \rho; \delta; \psi)$ if it satisfies

$$\frac{(1-\delta)\omega D_r^m(\theta,\vartheta,\rho)\varphi'(\omega)+\delta\omega(\omega D_r^m(\theta,\vartheta,\rho)\varphi'(\omega))'}{(1-\delta)D_r^m(\theta,\vartheta,\rho)\varphi_{\eta,\tau}(\omega)+\delta\omega D_r^m(\theta,\vartheta,\rho)\varphi'_{\eta,\tau}(\omega)} \prec \Lambda^*[\alpha,\beta;\psi(\omega)], \tag{1.12}$$

where $D_r^m(\theta, \vartheta, \rho)\varphi$ is defined as in (1.6) and $\psi \in \mathcal{P}$ is defined as in (1.2).

Remark 1.1. The defined class of functions involves lots of parameters, so we can obtain several classes as its special case. Here, we will present a few of them:

(1) If we let $\theta = \tau = m = 0$ and $\rho = 1$, the class $\mathcal{C}^m_{(\eta,\tau)}(\theta, \vartheta, \rho; \delta; \psi)$ will reduce to the classes $\mathcal{S}_s^{(\eta,\tau)}(\psi)$ and $\mathcal{C}_s^{(\eta,\tau)}(\psi)$ by choosing $\delta = 0$ and $\delta = 1$ respectively. The class $\mathcal{S}_s^{(\eta,\tau)}(\psi)$ and $\mathcal{C}_s^{(\eta,\tau)}(\psi)$ were recently introduced and studied by Karthikeyan in. (Karthikeyan 2013)
(2) If we let $\alpha = \beta = \frac{\pi}{2}$, $\theta = \tau = m = 0$, $\rho = 1$ and $\psi(\omega) = 1 + F\omega/1 + G\omega$ in Definition 1.1, then the class $\mathcal{C}^m_{(\eta,\tau)}(\theta, \vartheta, \rho; \delta; \psi)$ reduces to the class $\mathcal{S}^{(\eta,\tau)}[F, G]$ defined and studied by Al Sarari et al. [2016, Definition 5]
(3) If we let $\alpha = \beta = \frac{\pi}{2}$, $\theta = \tau = m = 0$, $\eta = \rho = 1$ and $\psi(\omega) = 1 + F\omega/1 + G\omega$ in Definition 1.1, then the class $\mathcal{C}^m_{(\eta,\tau)}(\theta, \vartheta, \rho; \delta; \psi)$ reduces to the class $\mathcal{S}_s^{(\tau)}[F, G]$ defined and studied by Kwon, and Sim. (Kwon and Sim 2013)

The analytic characterization of $\mathcal{C}^m_{(\eta,\tau)}(\theta, \vartheta, \rho; \delta; \psi)$ is very similar to the one that was defined by Karthikeyan et al. in (Karthikeyan et al. 2021). The major deviation here is that we have defined the class with respect to (η, τ)-symmetric points whereas the class studied by Karthikeyan et al. (2021) involved the odd function. The class $\mathcal{C}^m_{(\eta,\tau)}(\theta, \vartheta, \rho; \delta; \psi)$ neither unifies nor generalizes the study of Karthikeyan et al. (2021) Further, we have deviated in obtaining the results like subordination condition of starlikeness.

2. Coefficient inequalities

In this section, we impose another condition that the first coefficient of the series namely L_1 in (1.2) to be real. To obtain our main results, we need the following Lemma.

Lemma 2.1. *Let the function $\Lambda^*[\alpha, \beta; \psi(\omega)]$ be convex in $\mathbb{U}$ where the function ψ is defined as in* (1.2). *If $p(\omega) = 1 + \sum_{n=1}^{\infty} p_n\omega^n$ is analytic in $\mathbb{U}$ and satisfies the subordination condition*

$$p(\omega) \prec \Lambda^*[\alpha, \beta; \psi(\omega)], \tag{2.1}$$

then

$$|p_n| \le \sqrt{4\cos^2\beta + L_1^2\sin^2\alpha}, \ n \ge 1. \tag{2.2}$$

Proof. If the function ψ has the power series expansion (1.2), then

$$\Lambda^*[\alpha, \beta; \psi(\omega)] = 1 + e^{-i\alpha}(2\cos\beta + iL_1 \sin\alpha)\,\omega + \ldots,\ \ \omega \in \mathbb{U}.$$

The equation (2.1) is equivalent to

$$p(\omega) - 1 \prec \Lambda^*[\alpha, \beta; \psi(\omega)] - 1.$$

Since the convexity of $\Lambda^*[\alpha, \beta; \psi(\omega)]$ remains unaffected by translation, from a well-known Lemma of Rogosinski ([1943, Theorem VII]) it follows the conclusion (2.2). □

Here we present the coefficient inequality of $\mathcal{C}^m_{(\eta,\tau)}(\theta, \vartheta, \rho; \delta; \psi)$.

Theorem 2.2. *If $\varphi \in \mathcal{C}^m_{(\eta,\tau)}(\theta, \vartheta, \rho; \delta; \psi)$, then for $n \geq 2$,*

$$|a_n| \leq \frac{1}{[1 + \delta(n-1)]|\Phi_n|} \prod_{t=1}^{n-1} \frac{|\Gamma_{t,\eta}|\sqrt{4\cos^2\beta + L_1^2 \sin^2\alpha} + |t - \Gamma_{t,\eta}|}{|(t+1) - \Gamma_{t+1,\eta}|}, \tag{2.3}$$

where

$$\Phi_n = \left[n + \frac{r}{2}\left(1 + (-1)^{n+1}\right)\right]^m \frac{\Gamma(\vartheta)(\rho)_{n-1}}{\Gamma(\vartheta + \theta(n-1))(n-1)!}, n \geq 2$$

Proof. By the definition of $\mathcal{C}^m_{(\eta,\tau)}(\theta, \vartheta, \rho; \delta; \psi)$, we have

$$\frac{(1-\delta)\omega D_r^m(\theta, \vartheta, \rho)\varphi'(\omega) + \delta\omega(\omega D_r^m(\theta, \vartheta, \rho)\varphi'(\omega))'}{(1-\delta)D_r^m(\theta, \vartheta, \rho)\varphi_{\eta,\tau}(\omega) + \delta\omega D_r^m(\theta, \vartheta, \rho)\varphi'_{\eta,\tau}(\omega)} = p(\omega), \tag{2.4}$$

where $p(\omega) \in \mathcal{P}$ is subordinate to $p(\omega) \prec \Lambda^*[\alpha, \beta; \psi(\omega)]$.

With $a_1 = \Phi_1 = \Gamma_{1,\eta} = 1$, (2.4) can be written as

$$(1 - \Gamma_{1,\eta})\omega + \sum_{n=2}^{\infty}(n - \Gamma_{n,\eta})[1 + \delta(n-1)]\Phi_n a_n \omega^n$$
$$= \left(\sum_{n=1}^{\infty} p_n \omega^n\right)\left(\sum_{n=1}^{\infty}[1 + \delta(n-1)]\Gamma_{n,\eta}\Phi_n a_n \omega^n\right).$$

From the above equality, we have

$$(n - \Gamma_{n,\eta})[1 + \delta(n-1)]\Phi_n a_n = [\Gamma_{n-1,\eta}[1 + \delta(n-2)]\Phi_{n-1}a_{n-1}p_1 + \cdots + p_{n-1}\Gamma_{1,\eta}\Phi_1 a_1]$$

$$= \sum_{t=1}^{n-1}\left|p_{n-t}[1 + (t-1)\delta]\Gamma_{t,\eta}\Phi_t a_t\right| \leq \sum_{t=1}^{n-1}(1 + \delta(t-1))|p_{n-t}\Gamma_{t,\eta}|\Phi_t|a_t|.$$

The assertion of 2.1 implies $|p_n| \leq \sqrt{4\cos^2\beta + L_1^2\sin^2\alpha}$, $n \geq 1$. On computation, we have

$$|a_n| \le \frac{\sqrt{4\cos^2\beta + L_1^2\sin^2\alpha}\sum_{t=1}^{n-1}(1+\delta(t-1))|\Phi_t\Gamma_{t,\eta}||a_t|}{\left|n-\Gamma_{n,\eta}\right|[1+\delta(n-1)]|\Phi_n|}. \tag{2.5}$$

Let $n = 2$ in (2.5), then

$$|a_2| \le \frac{\sqrt{4\cos^2\beta + L_1^2\sin^2\alpha}}{\left|2-\Gamma_{2,\eta}\right|[1+\delta]|\Phi_2|}. \tag{2.6}$$

Letting $n = 2$ in (2.3), we get

$$\begin{aligned}|a_2| &\le \frac{1}{[1+\delta]|\Phi_2|}\prod_{t=1}^{2-1}\frac{\left|\Gamma_{t,\eta}\right|\sqrt{4\cos^2\beta + L_1^2\sin^2\alpha}+\left|t-\Gamma_{t,\eta}\right|}{\left|(t+1)-\Gamma_{t+1,\eta}\right|}\\ &= \frac{1}{[1+\delta]|\Phi_2|}\frac{\sqrt{4\cos^2\beta + L_1^2\sin^2\alpha}}{\left|2-\Gamma_{2,\eta}\right|}.\end{aligned} \tag{2.7}$$

From (4.2) and (2.7), we conclude that (2.3) is correct for $n = 2$. Now let $n = 3$ in (2.5), we have

$$|a_3| \le \frac{\sqrt{4\cos^2\beta + L_1^2\sin^2\alpha}}{\left|3-\Gamma_{3,\eta}\right|[1+2\delta]|\Phi_3|}\left[\left|\Gamma_{1,\eta}\right| + (1+2\delta)\left|\Gamma_{2,\eta}\right|\Phi_2|a_2|\right]$$

$$\le \frac{\sqrt{4\cos^2\beta + L_1^2\sin^2\alpha}}{\left|3-\Gamma_{3,\eta}\right|[1+2\delta]|\Phi_3|}\left[1+\frac{\left|\Gamma_{2,\eta}\right|\sqrt{4\cos^2\beta + L_1^2\sin^2\alpha}}{\left|2-\Gamma_{2,\eta}\right|}\right].$$

If we let $n = 3$, in (2.3), we have

$$\begin{aligned}|a_3| &\le \frac{1}{[1+2\delta]|\Phi_3|}\left[\frac{\sqrt{4\cos^2\beta + L_1^2\sin^2\alpha}}{\left|2-\Gamma_{2,\eta}\right|}\times\frac{\left|\Gamma_{2,\eta}\right|\sqrt{4\cos^2\beta + L_1^2\sin^2\alpha}+\left|2-\Gamma_{2,\eta}\right|}{\left|3-\Gamma_{3,\eta}\right|}\right]\\ &\le \frac{1}{[1+2\delta]|\Phi_3|}\left[\frac{\sqrt{4\cos^2\beta + L_1^2\sin^2\alpha}}{\left|3-\Gamma_{3,\eta}\right|}\times\frac{\left|\Gamma_{2,\eta}\right|\sqrt{4\cos^2\beta + L_1^2\sin^2\alpha}+\left|2-\Gamma_{2,\eta}\right|}{\left|2-\Gamma_{2,\eta}\right|}\right]\\ &\le \frac{\sqrt{4\cos^2\beta + L_1^2\sin^2\alpha}}{[1+2\delta]|\Phi_3|\left|3-\Gamma_{3,\eta}\right|}\left[1+\frac{\left|\Gamma_{2,\eta}\right|\sqrt{4\cos^2\beta + L_1^2\sin^2\alpha}}{\left|2-\Gamma_{2,\eta}\right|}\right].\end{aligned}$$

Hence the hypothesis is correct for $n = 3$. Assume that (2.3) is valid for $n = 2, 3, \ldots r$. So from (2.3), we have

$$|a_r| \le \frac{1}{[1+\delta(r-1)]|\Phi_r|}\prod_{t=1}^{r-1}\frac{\left|\Gamma_{t,\eta}\right|\sqrt{4\cos^2\beta+L_1^2\sin^2\alpha}+\left|t-\Gamma_{t,\eta}\right|}{\left|(t+1)-\Gamma_{t+1,\eta}\right|}.$$

By induction hypothesis, we have

$$\begin{aligned}|a_r| &\leq \frac{1}{[1+\delta(r-1)]|\Phi_r|}\prod_{t=1}^{r-1}\frac{|\Gamma_{t,\eta}|\sqrt{4\cos^2\beta+L_1^2\sin^2\alpha}+|t-\Gamma_{t,\eta}|}{|(t+1)-\Gamma_{t+1,\eta}|}\\ &= \frac{\sqrt{4\cos^2\beta+L_1^2\sin^2\alpha}}{|r-\Gamma_{r,\eta}|[1+\delta(r-1)]|\Phi_r|}\sum_{t=1}^{r-1}(1+\delta(t-1))|\Phi_t\Gamma_{t,\eta}||a_t|.\end{aligned} \tag{2.8}$$

Now letting $n = r + 1$ in (2.5), we have

$$\begin{aligned}|a_{r+1}| &\leq \frac{\sqrt{4\cos^2\beta+L_1^2\sin^2\alpha}}{|(r+1)-\Gamma_{r+1,\eta}|[1+\delta(r)]|\Phi_{r+1}|}\sum_{t=1}^{r}(1+\delta(t-1))|\Phi_t\Gamma_{t,\eta}||a_t|\\ &= \frac{\sqrt{4\cos^2\beta+L_1^2\sin^2\alpha}}{|(r+1)-\Gamma_{r+1,\eta}|[1+\delta(r)]|\Phi_{r+1}|}\\ &\quad\left[\sum_{t=1}^{r-1}(1+\delta(t-1))|\Phi_t\Gamma_{t,\eta}||a_t|+(1+\delta(r-1))|\Phi_r\Gamma_{r,\eta}||a_r|\right]\end{aligned} \tag{2.9}$$

Using (2.8) in (2.9), we can obtain

$$\begin{aligned}|a_{r+1}| &\leq \frac{\sqrt{4\cos^2\beta+L_1^2\sin^2\alpha}}{|(r+1)-\Gamma_{r+1,\eta}|[1+\delta(r)]|\Phi_{r+1}|}\left[\sum_{t=1}^{r-1}(1+\delta(t-1))|\Phi_t\Gamma_{t,\eta}||a_t|\right.\\ &\quad\left.+\frac{|\Gamma_{r,\eta}|\sqrt{4\cos^2\beta+L_1^2\sin^2\alpha}}{|r-\Gamma_{r,\eta}|}\sum_{t=1}^{r-1}(1+\delta(t-1))|\Phi_t\Gamma_{t,\eta}||a_t|\right]\\ &= \frac{\sqrt{4\cos^2\beta+L_1^2\sin^2\alpha}\sum^{t=1} r-1(1+\delta(t-1))|\Phi_t\Gamma_{t,\eta}||a_t|}{|(r+1)-\Gamma_{r+1,\eta}|[1+\delta(r)]|\Phi_{r+1}|}\\ &\quad\left[\frac{|\Gamma_{r,\eta}|\sqrt{4\cos^2\beta+L_1^2\sin^2\alpha}+|r-\Gamma_{r,\eta}|}{|r-\Gamma_{r,\eta}|}\right]\\ &= \frac{\sqrt{4\cos^2\beta+L_1^2\sin^2\alpha}\sum^{t=1} r-1(1+\delta(t-1))|\Phi_t\Gamma_{t,\eta}||a_t|}{|r-\Gamma_{r,\eta}|[1+\delta(r)]|\Phi_{r+1}|}\\ &\quad\left[\frac{|\Gamma_{r,\eta}|\sqrt{4\cos^2\beta+L_1^2\sin^2\alpha}+|r-\Gamma_{r,\eta}|}{|(r+1)-\Gamma_{r+1,\eta}|}\right]\\ &= \frac{1}{[1+\delta(r)]|\Phi_{r+1}|}\prod_{t=1}^{r-1}\frac{|\Gamma_{t,\eta}|\sqrt{4\cos^2\beta+L_1^2\sin^2\alpha}+|t-\Gamma_{t,\eta}|}{|(t+1)-\Gamma_{t+1,\eta}|}\times\\ &\quad\left[\frac{|\Gamma_{r,\eta}|\sqrt{4\cos^2\beta+L_1^2\sin^2\alpha}+|r-\Gamma_{r,\eta}|}{|(r+1)-\Gamma_{r+1,\eta}|}\right]\\ &= \frac{1}{[1+\delta(r)]|\Phi_{r+1}|}\prod_{t=1}^{r}\frac{|\Gamma_{t,\eta}|\sqrt{4\cos^2\beta+L_1^2\sin^2\alpha}+|t-\Gamma_{t,\eta}|}{|(t+1)-\Gamma_{t+1,\eta}|},\end{aligned}$$

implies that inequality (2.3) is true for $n = r + 1$. Hence, the proof of the Theorem. □

If we let $\alpha = \beta = \frac{\pi}{2}$, $\theta = \tau = m = 0$, $\lambda = \rho = 1$ and $\psi(\omega) = (1 + F\omega)/(1 + G\omega)$ in Theorem 2.2, then we get the following result.

Corollary 2.3. [Al Sarari et al. 2016, Theorem 2] *If $\varphi \in \mathcal{S}^{(\eta,\tau)}(F, G)$ (see Remark 1.1), then for $n \geq 2$, $-1 \leq F < G \leq 1$,*

$$|a_n| \leq \prod_{t=1}^{n-1} \frac{|\Gamma_{t,\eta}|[(F-G)-1]+t}{\left|t+1-\Gamma_{t+1,\eta}\right|}.$$

For $F = 1 - 2\xi \cos \varrho$, $0 \leq \xi < 1$, $G = -1$ and $\eta = 1$, the Corollary 2.3 reduces to the next special case:

Corollary 2.4. [Libera. 1967, Theorem 1] *If $\varphi \in \mathcal{A}$ satisfy the inequality*

$$\text{Re}\,\frac{\omega\varphi'(\omega)}{\varphi(\omega)} > \xi \cos \varrho,\ \ \omega \in \mathbb{U},$$

then

$$|a_n| \leq \prod_{t=0}^{n-2} \frac{|2(1-\xi)e^{-i\rho}\cos\varrho + t|}{t+1}. \tag{2.10}$$

The coefficient estimates of (2.10) are sharp

If we let $\alpha = \beta = \frac{\pi}{2}$, $\theta = \tau = m = 0$, $\lambda = \rho = 1$ and $\psi(\omega) = \frac{(F+1)p_\alpha(\omega)-(F-1)}{(G+1)p_\alpha(\omega)-(G-1)}$ in Theorem 2.2, then we get the following result.

Corollary 2.5. [Karthikeyan et al. 2020, Corollary 6] *If $\varphi \in \mathcal{A}$ satisfy the condition*

$$\frac{\omega\varphi'(\omega)}{\varphi(\omega)} \prec \frac{(F+1)p_\alpha(\omega)-(F-1)}{(G+1)p_\alpha(\omega)-(G-1)},$$

with $p_\alpha(\omega) = (1+2\alpha)\sqrt{\frac{1+b\omega}{1-bz}} - 2\alpha$, $b = b(\alpha) = \frac{1+4\alpha-4\alpha^2}{(1+2\alpha)^2}$, $\alpha > 0$. *Then for* $n \geq 2$, $-1 \leq H < G \leq 1$,

$$|a_n| \leq \prod_{t=0}^{n-2} \frac{|(F-G)(1+4\alpha)-2tG|}{2(1+2\alpha)(t+1)}.$$

Remark 2.1. Several well-known results can be obtained as special case of Theorem 2.2.

3. Inclusion relationships and integral representations of the classes $\mathcal{C}^m_{(\eta,\tau)}(\theta, \vartheta, \rho; \delta; \psi)$

If $\mathcal{C}^m_{(\eta,\tau)}(\theta, \vartheta, \rho; \delta; \psi)$, then by Definition 1.1 there exist a Schwatrz function $\sigma(\omega)$ such that

$$\frac{(1-\delta)\omega D^m_r(\theta, \vartheta, \rho)\varphi'(\omega) + \delta\omega(\omega D^m_r(\theta, \vartheta, \rho)\varphi'(\omega))'}{(1-\delta)D^m_r(\theta, \vartheta, \rho)\varphi_{\eta,\tau}(\omega) + \delta\omega D^m_r(\theta, \vartheta, \rho)\varphi'_{\eta,\tau}(\omega)} = \Lambda^*[\alpha, \beta; \psi(\sigma(\omega))]. \tag{3.1}$$

If we replace ω by $\varepsilon^\nu\omega\,(\nu = 0, 1, 2, \ldots, \tau-1)$ in (3.1), then (3.1) will be of the form

$$\frac{(1-\delta)\varepsilon^\nu\omega D^m_r(\theta, \vartheta, \rho)\varphi'(\varepsilon^\nu\omega) + \delta\varepsilon^\nu\omega(\varepsilon^\nu\omega D^m_r(\theta, \vartheta, \rho)\varphi'(\varepsilon^\nu\omega))'}{(1-\delta)D^m_r(\theta, \vartheta, \rho)\varphi_{\eta,\tau}(\varepsilon^\nu\omega) + \delta\varepsilon^\nu\omega D^m_r(\theta, \vartheta, \rho)\varphi'_{\eta,\tau}(\varepsilon^\nu\omega)} = \Lambda^*[\alpha, \beta; \psi(\sigma(\varepsilon^\nu\omega))]. \tag{3.2}$$

Using (1.9) in (3.2), we get

$$\frac{(1-\delta)\varepsilon^\nu\omega D^m_r(\theta, \vartheta, \rho)\varphi'(\varepsilon^\nu\omega) + \delta\varepsilon^\nu\omega(\varepsilon^\nu\omega D^m_r(\theta, \vartheta, \rho)\varphi'(\varepsilon^\nu\omega))'}{\varepsilon^{\nu\eta}\left[(1-\delta)D^m_r(\theta, \vartheta, \rho)\varphi_{\eta,\tau}(\omega) + \delta\omega D^m_r(\theta, \vartheta, \rho)\varphi'_{\eta,\tau}(\omega)\right]} = \Lambda^*[\alpha, \beta; \psi(\sigma(\varepsilon^\nu\omega))]. \tag{3.3}$$

Let $\nu = 0, 1, 2, \ldots, \tau-1$ in (3.3) respectively and summing them, we get

$$\frac{(1-\delta)\frac{1}{\tau}\sum_{\nu=0}^{\tau-1}\varepsilon^{\nu-\nu\eta}\omega D^m_r(\theta, \vartheta, \rho)\varphi'(\varepsilon^\nu\omega) + \delta\left[\frac{1}{\tau}\sum_{\nu=0}^{\tau-1}\varepsilon^{\nu-\nu\eta}\omega D^m_r(\theta, \vartheta, \rho)\varphi'(\varepsilon^\nu\omega) + \frac{1}{\tau}\sum_{\nu=0}^{\tau-1}\varepsilon^{2\nu-\nu\eta}\omega^2 D^m_r(\theta, \vartheta, \rho)\varphi(\varepsilon^\nu\omega)\right]}{\left[(1-\delta)D^m_r(\theta, \vartheta, \rho)\varphi_{\eta,\tau}(\omega) + \delta\omega D^m_r(\theta, \vartheta, \rho)\varphi'_{\eta,\tau}(\omega)\right]}$$
$$= \frac{1}{\tau}\sum_{\nu=0}^{\tau-1}\Lambda^*[\alpha, \beta; \psi(\sigma(\varepsilon^\nu\omega))].$$

Or equivalently,

$$\frac{(1-\delta)\omega D^m_r(\theta, \vartheta, \rho)\varphi'_{\eta,\tau}(\omega) + \delta\omega(\omega D^m_r(\theta, \vartheta, \rho)\varphi'_{\eta,\tau}(\omega))'}{(1-\delta)D^m_r(\theta, \vartheta, \rho)\varphi_{\eta,\tau}(\omega) + \delta\omega D^m_r(\theta, \vartheta, \rho)\varphi'_{\eta,\tau}(\omega)} = \frac{1}{\tau}\sum_{\nu=0}^{\tau-1}\Lambda^*[\alpha, \beta; \psi(\sigma(\varepsilon^\nu\omega))].$$

On summarizing the above discussion, we have the following.

Theorem 3.1. Let the function $\Lambda^*[\alpha, \beta; \psi(\omega)] \in \mathcal{NP}$ satisfy the subordination condition $\frac{1}{\tau}\sum_{\nu=0}^{\tau-1}\Lambda^*[\alpha, \beta; \psi(\sigma(\varepsilon^\nu\omega))] \prec \Lambda^*[\alpha, \beta; \psi(\omega)]$. If $\varphi \in \mathcal{C}^m_{(\eta,\tau)}(\theta, \vartheta, \rho; \delta; \psi)$, then

$$\frac{(1-\delta)\omega D^m_r(\theta, \vartheta, \rho)\varphi'_{\eta,\tau}(\omega) + \delta\omega(\omega D^m_r(\theta, \vartheta, \rho)\varphi'_{\eta,\tau}(\omega))'}{(1-\delta)D^m_r(\theta, \vartheta, \rho)\varphi_{\eta,\tau}(\omega) + \delta\varepsilon^\nu\omega D^m_r(\theta, \vartheta, \rho)\varphi'_{\eta,\tau}(\omega)} \prec \Lambda^*[\alpha, \beta; \psi(\omega)].$$

Let $G(\omega) = (1-\delta)D_r^m(\theta, \vartheta, \rho)\varphi(\omega) + \delta\omega D_r^m(\theta, \vartheta, \rho)\varphi'(\omega)$ and $H(\omega) = (1-\delta)D_r^m(\theta, \vartheta, \rho)\varphi_{\eta,\tau}(\omega) + \delta\omega D_r^m(\theta, \vartheta, \rho)\varphi'_{\eta,\tau}(\omega)$. If $\varphi \in \mathcal{C}^m_{(\eta,\tau)}(\theta, \vartheta, \rho; \delta; \psi)$, then following the steps as in Theorem 3.1, we have

$$\frac{\omega H'(\omega)}{H(\omega)} = \frac{1}{\tau}\sum_{\nu=0}^{\tau-1} \Lambda^*[\alpha, \beta; \psi(\sigma(\varepsilon^\nu \omega))].$$

Alternatively, the above equality can be rewritten as

$$\frac{H'(\omega)}{H(\omega)} - \frac{1}{\omega} = \frac{1}{\tau}\sum_{\nu=0}^{\tau-1} \frac{\Lambda^*[\alpha, \beta; \psi(\sigma(\varepsilon^\nu \omega))] - 1}{\omega}.$$

Integrating this equality, we get

$$\begin{aligned}\log\left\{\frac{H(\omega)}{\omega}\right\} &= \frac{1}{\tau}\sum_{\nu=0}^{\tau-1}\int_0^{\omega} \frac{\Lambda^*[\alpha, \beta; \psi(\sigma(\varepsilon^\nu \zeta))] - 1}{\zeta}\, d\zeta \\ &= \frac{1}{\tau}\sum_{\nu=0}^{\tau-1}\int_0^{\varepsilon^\nu \omega} \frac{\Lambda^*[\alpha, \beta; \psi(\sigma(t))] - 1}{t}\, dt,\end{aligned}$$

or equivalently,

$$\begin{aligned}&(1-\delta)D_r^m(\theta, \vartheta, \rho)\varphi_{\eta,\tau}(\omega) + \delta\omega D_r^m(\theta, \vartheta, \rho)\varphi'_{\eta,\tau}(\omega) \\ &= \omega \exp\left\{\frac{1}{\tau}\sum_{\nu=0}^{\tau-1}\int_0^{\varepsilon^\nu \omega} \frac{\Lambda^*[\alpha, \beta; \psi(\sigma(t))] - 1}{t}\, dt\right\}.\end{aligned}$$

We have two cases namely

(1) For $\delta = 0$, trivially we have

$$D_r^m(\theta, \vartheta, \rho)\varphi_{\eta,\tau}(\omega) = \omega \exp\left\{\frac{1}{\tau}\sum_{\nu=0}^{\tau-1}\int_0^{\varepsilon^\nu \omega} \frac{\Lambda^*[\alpha, \beta; \psi(\sigma(t))] - 1}{t}\, dt\right\}.$$

(2) For $0 < \delta \le 1$,

$$D_r^m(\theta, \vartheta, \rho)\varphi_{\eta,\tau}(\omega) = \frac{1}{\delta}\omega^{\left(1-\frac{1}{\delta}\right)}\int_0^{\omega} u^{\frac{1}{\delta}-1} \exp\left\{\frac{1}{\tau}\sum_{\nu=0}^{\tau-1}\int_0^{\varepsilon^\nu \omega} \frac{\Lambda^*[\alpha, \beta; \psi(\sigma(t))] - 1}{t}\, dt\right\} du.$$

Summarising the above discussion, we have

Theorem 3.2. *If $\varphi \in \mathcal{C}^m_{(\eta,\tau)}(\theta, \vartheta, \rho; \delta; \psi)$, then*

(i) *for $0 < \delta \le 1$,*

$$D_r^m(\theta, \vartheta, \rho)\varphi_{\eta,\tau}(\omega) = \frac{1}{\delta}\omega^{\left(1-\frac{1}{\delta}\right)}\int_0^{\omega} u^{\frac{1}{\delta}-1} \exp\left\{\frac{1}{\tau}\sum_{\nu=0}^{\tau-1}\int_0^{\varepsilon^\nu \omega} \frac{\Lambda^*[\alpha, \beta; \psi(\sigma(t))] - 1}{t}\, dt\right\} du. \tag{3.4}$$

(ii) *for* $\delta = 0$,

$$D_r^m(\theta, \vartheta, \rho)\varphi_{\eta,\tau}(\omega) = \omega \exp\left\{\frac{1}{\tau}\sum_{\nu=0}^{\tau-1}\int_0^{\varepsilon^\nu \omega}\frac{\Lambda^*[\alpha, \beta; \psi(\sigma(t))] - 1}{t}\,dt\right\}. \tag{3.5}$$

If we let $\theta = \tau = m = \delta = 0$, $\lambda = \rho = 1$ and $\alpha = \beta = \frac{\pi}{2}$ in Theorem 3.2, we get

Corollary 3.3. [13, Theorem 2.3] *Let* $\varphi \in \mathcal{S}_s^{(\eta,\tau)}(\psi)$, *then we have*

$$\varphi_{\eta,\tau}(\omega) = \omega \exp\left\{\frac{1}{\tau}\sum_{\nu=0}^{\tau-1}\int_0^{\varepsilon^\nu \omega}\frac{\psi(\sigma(t)) - 1}{t}\,dt\right\}$$

where $\varphi_{\eta,\tau}(\omega)$ *defined by equality (1.8),* $\sigma(\omega)$ *is analytic in* $\mathbb{U}$ *and* $\sigma(0) = 0$, $|\sigma(\omega)| < 1$.

If we let $\theta = \tau = m = 0$, $\delta = \lambda = \rho = 1$ and $\alpha = \beta = \frac{\pi}{2}$ in Theorem 3.2, we have the following Corollary.

Corollary 3.4. [13, Theorem 2.4] *Let* $\varphi \in \mathcal{C}_s^{(\eta,\tau)}(\psi)$, *then we have*

$$\varphi_{\eta,\tau}(\omega) = \int_0^\omega \exp\left\{\frac{1}{\tau}\sum_{\nu=0}^{\tau-1}\int_0^{\varepsilon^\nu \zeta}\frac{\psi(\sigma(t)) - 1}{t}\,dt\right\}d\zeta$$

where $\varphi_{\eta,\tau}(\omega)$ *defined by equality (1.8),* $\sigma(\omega)$ *is analytic in* $\mathbb{U}$ *and* $\sigma(0) = 0$, $|\sigma(\omega)| < 1$.

4. Subordination conditions for the classes $\mathcal{C}^m_{(\eta,\tau)}(\theta, \vartheta, \rho; \delta; \psi)$

Note that the function $\Lambda^*[\alpha, \beta; \psi(\omega)]$ in general does not belong to the class $\mathcal{P}$ and is not convex. However, if we restrict the radius of the domain or by choosing appropriate values of the parameter, we can see that $\Lambda^*[\alpha, \beta; \psi(\omega)]$ will belong to class $\mathcal{P}$.

Motivated by the results presented in Chapter 4 of (Bulboacă 2005), here we obtain some conditions for starlikeness. We now state the following result which will be used in the sequel.

Lemma 4.1 ([8]). *Let g be convex in* $\mathbb{U}$, *with* $g(0) = a$, $\gamma \neq 0$ *and* $\mathrm{Re}(\gamma) > 0$. *Suppose that* $h(\omega)$ *is analytic in* $\mathbb{U}$, *which is given by*

$$h(\omega) = a + \vartheta_n\omega^n + \vartheta_{n+1}\omega^{n+1} + \cdots, \quad \omega \in \mathbb{U}. \tag{4.1}$$

If

$$h(\omega) + \frac{\omega h'(\omega)}{\gamma} \prec g(\omega),$$

then

$$h(\omega) \prec q(\omega) \prec g(\omega),$$

where

$$q(\omega)=\frac{\gamma}{n\,\omega^{\gamma/n}}\int_0^{\omega}g(t)\,t^{(\gamma/n)-1}dt.$$

The function q is convex and is the best (a, n)-dominant.

For convenience, we denote $G(\omega)=(1-\delta)D_r^m(\theta,\vartheta,\rho)\varphi(\omega)+\delta\omega D_r^m(\theta,\vartheta,\rho)\varphi'(\omega)$ and $H(\omega)=(1-\delta)D_r^m(\theta,\vartheta,\rho)\varphi_{\eta,\tau}(\omega)+\delta\omega D_r^m(\theta,\vartheta,\rho)\varphi'_{\eta,\tau}(\omega)$.

Theorem 4.2. Let the function $\Lambda^*[\alpha,\beta;p(\omega)]$ be defined as in (1.4) be convex univalent in $\mathbb{U}$. Let $\varphi\in\mathcal{A}$ satisfy

$$\frac{\omega G'(\omega)}{H(\omega)}\left[1+\frac{\omega D_r^m(\theta,\vartheta,\rho)\varphi'(\omega)+(2\delta+1)\omega^2D_r^m(\theta,\vartheta,\rho)\varphi''(\omega)+\delta\omega^3D_r^m(\theta,\vartheta,\rho)\varphi'''(\omega)}{\omega D_r^m(\theta,\vartheta,\rho)\varphi'(\omega)+\delta\omega^2D_r^m(\theta,\vartheta,\rho)\varphi''(\omega)}\right.$$
$$\left.-\frac{\omega D_r^m(\theta,\vartheta,\rho)\varphi'_{\eta,\tau}(\omega)+\delta\omega^2D_r^m(\theta,\vartheta,\rho)\varphi''_{\eta,\tau}(\omega)}{(1-\delta)D_r^m(\theta,\vartheta,\rho)\varphi_{\eta,\tau}(\omega)+\delta\omega D_r^m(\theta,\vartheta,\rho)\varphi'_{\eta,\tau}(\omega)}\right]\prec\Lambda^*[\alpha,\beta;\psi(\omega)], \tag{4.2}$$

then

$$\frac{(1-\delta)\omega D_r^m(\theta,\vartheta,\rho)\varphi'(\omega)+\delta\omega(\omega D_\tau^m(\theta,\vartheta,\rho)\varphi'(\omega))'}{(1-\delta)D_r^m(\theta,\vartheta,\rho)\varphi_{\eta,\tau}(\omega)+\delta\omega D_r^m(\theta,\vartheta,\rho)\varphi'_{\eta,\tau}(\omega)}\prec q(\omega)$$
$$=\frac{1}{\omega}\int_0^{\omega}\left(\frac{e^{-i\alpha}[\cos\alpha+i\sin\alpha(1-t^2)\psi(t)]}{1-2\cos\beta te^{-i\alpha}+e^{-2i\alpha}t^2}\right)dt\prec\Lambda^*[\alpha,\beta;\psi(\omega)]. \tag{4.3}$$

and $q(\omega)$ is the best dominant.

Proof. Let $p(\omega)$ be defined by

$$p(\omega)=\frac{(1-\delta)\omega D_r^m(\theta,\vartheta,\rho)\varphi'(\omega)+\delta\omega(\omega D_\tau^m(\theta,\vartheta,\rho)\varphi'(\omega))'}{(1-\delta)D_r^m(\theta,\vartheta,\rho)\varphi_{\eta,\tau}(\omega)+\delta\omega D_r^m(\theta,\vartheta,\rho)\varphi'_{\eta,\tau}(\omega)}=\frac{\omega G'(\omega)}{H(\omega)},\quad \omega\in\mathbb{U}. \tag{4.4}$$

Then the function $p(\omega)$ is of the form $p(\omega)=1+p_1\omega+p_2\omega^2+\cdots$ and is analytic in $\mathbb{U}$. Differentiating both sides of (4.4) and by simplifying, we have

$$\frac{\omega G'(\omega)}{H(\omega)}\left[1+\frac{\omega D_r^m(\theta,\vartheta,\rho)\varphi'(\omega)+(2\delta+1)\omega^2D_r^m(\theta,\vartheta,\rho)\varphi''(\omega)+\delta\omega^3D_r^m(\theta,\vartheta,\rho)\varphi'''(\omega)}{\omega D_r^m(\theta,\vartheta,\rho)\varphi'(\omega)+\delta\omega^2D_r^m(\theta,\vartheta,\rho)\varphi''(\omega)}\right.$$
$$\left.-\frac{\omega D_r^m(\theta,\vartheta,\rho)\varphi'_{\eta,\tau}(\omega)+\delta\omega^2D_r^m(\theta,\vartheta,\rho)\varphi''_{\eta,\tau}(\omega)}{(1-\delta)D_r^m(\theta,\vartheta,\rho)\varphi_{\eta,\tau}(\omega)+\delta\omega D_r^m(\theta,\vartheta,\rho)\varphi'_{\eta,\tau}(\omega)}\right]=p(\omega)+\omega p'(\omega). \tag{4.5}$$

By hypothesis (4.2), we have

$$p(\omega)+\omega p'(\omega)\prec\frac{e^{-i\alpha}[\cos\alpha+i\sin\alpha(1-\omega^2)\psi(\omega)]}{1-2\cos\beta\omega e^{-i\alpha}+e^{-2i\alpha}\omega^2}.$$

Applying Lemma 4.1 to the above equation with $\gamma = 1$ and $a = n = 1$, we get the assertion (4.2) Hence, the proof of the Theorem 4.2 □

Remark 4.1. In Lemma 4.1, there is no need for the superordinate function to be in class $\mathcal{P}$. Hence, the choice of $\Lambda^*[\alpha, \beta; \psi(\omega)] \in \mathcal{NP}$ is admissible.

Theorem 4.3. Let the function $\Lambda^*[\alpha, \beta; \psi(\omega)] \in \mathcal{P}$ be convex univalent in $\mathbb{U}$ and let $\kappa(\omega) := (\Lambda^*[\alpha, \beta; \psi(\omega)])^2 + \omega(\Lambda^*[\alpha, \beta; \psi(\omega)])'$. If the function $\varphi \in A$ satisfies the conditions

$$\frac{\omega G'(\omega)}{H(\omega)} \times \left[\frac{\omega D_r^m(\theta, \vartheta, \rho)\varphi'(\omega) + (2\delta + 1)\omega^2 D_r^m(\theta, \vartheta, \rho)\varphi''(\omega) + \delta\omega^3 D_r^m(\theta, \vartheta, \rho)\varphi'''(\omega)}{\omega D_r^m(\theta, \vartheta, \rho)\varphi'(\omega) + \delta\omega^2 D_r^m(\theta, \vartheta, \rho)\varphi''(\omega)}\right. \tag{4.6}$$

$$\left. - \frac{\omega D_r^m(\theta, \vartheta, \rho)\varphi'_{\eta,\tau}(\omega) + \delta\omega^2 D_r^m(\theta, \vartheta, \rho)\varphi''_{\eta,\tau}(\omega)}{(1-\delta)D_r^m(\theta, \vartheta, \rho)\varphi_{\eta,\tau}(\omega) + \delta\omega D_r^m(\theta, \vartheta, \rho)\varphi'_{\eta,\tau}(\omega)} + \frac{\omega G'(\omega)}{H(\omega)}\right] \prec \kappa(\omega), \tag{4.7}$$

then $\varphi \in \mathcal{C}^m_{(\eta,\tau)}(\theta, \vartheta, \rho; \delta; \psi)$. Moreover, the function $\Lambda^*[\alpha, \beta; \psi(\omega)]$ is the best dominant of the left-hand side of (1.12).

Proof. If we define the function $p(\omega)$ by

$$p(\omega) := \frac{(1-\delta)\omega D_r^m(\theta, \vartheta, \rho)\varphi'(\omega) + \delta\omega(\omega D_\tau^m(\theta, \vartheta, \rho)\varphi'(\omega))'}{(1-\delta)D_r^m(\theta, \vartheta, \rho)\varphi_{\eta,\tau}(\omega) + \delta\omega D_r^m(\theta, \vartheta, \rho)\varphi'_{\eta,\tau}(\omega)}, \quad \omega \in \mathbb{U},$$

then from the hypothesis, it follows that p is analytic in $\mathbb{U}$. By a straight forward computation, we have

$$\omega p'(\omega) = p(\omega)\left[\frac{\omega D_r^m(\theta, \vartheta, \rho)\varphi'(\omega) + (2\delta + 1)\omega^2 D_r^m(\theta, \vartheta, \rho)\varphi''(\omega) + \delta\omega^3 D_r^m(\theta, \vartheta, \rho)\varphi'''(\omega)}{\omega D_r^m(\theta, \vartheta, \rho)\varphi'(\omega) + \delta\omega^2 D_r^m(\theta, \vartheta, \rho)\varphi''(\omega)}\right.$$
$$\left. - \frac{\omega D_r^m(\theta, \vartheta, \rho)\varphi'_{\eta,\tau}(\omega) + \delta\omega^2 D_r^m(\theta, \vartheta, \rho)\varphi''_{\eta,\tau}(\omega)}{(1-\delta)D_r^m(\theta, \vartheta, \rho)\varphi_{\eta,\tau}(\omega) + \delta\omega D_r^m(\theta, \vartheta, \rho)\varphi'_{\eta,\tau}(\omega)}\right],$$

and thus, the subordination (4.7) is equivalent to

$$p^2(\omega) + \omega p'(\omega) \prec \kappa(\omega). \tag{4.8}$$

Setting $\Omega(\sigma) := \sigma^2$ and $\Upsilon(\sigma) := 1$, then Ω and Υ are analytic functions in $\mathbb{C}$, with $\Upsilon(0) \neq 0$. Therefore

$$Q(\omega) = \omega(\Lambda^*[\alpha, \beta; \psi(\omega)])'\Upsilon(\Lambda^*[\alpha, \beta; \psi(\omega)]) = \omega(\Lambda^*[\alpha, \beta; \psi(\omega)])'$$

and

$$\kappa(\omega) = \Omega(\Lambda^*[\alpha, \beta; \psi(\omega)]) + Q(\omega) = (\Lambda^*[\alpha, \beta; \psi(\omega)])^2 + \omega(\Lambda^*[\alpha, \beta; \psi(\omega)])',$$

and using the assumption that $\Lambda^*[\alpha, \beta; \psi(\omega)]$ is a convex univalent function in $\mathbb{U}$, it follows that

$$\Re\frac{\omega Q'(\omega)}{Q(\omega)} = \Re\left(1 + \frac{\omega(\Lambda^*[\alpha, \beta; \psi(\omega)])''}{(\Lambda^*[\alpha, \beta; \psi(\omega)])'}\right) > 0, \ \omega \in \mathbb{U},$$
$$\left(Q'(0) = \left[(\Lambda^*[\alpha, \beta; \psi(\omega)])'\right]_{t=0} \neq 0\right),$$

hence Q is a starlike univalent function in $\mathbb{U}$. Further, the convexity of $\Lambda^*[\alpha, \beta; \psi(\omega)]$ together with $\Re[\Lambda^*[\alpha, \beta; \psi(\omega)]] > 0$ (assumed) implies

$$\Re\frac{\omega\kappa'(\omega)}{Q(\omega)} = \Re\left\{2\Lambda^*[\alpha, \beta; \psi(\omega)] + \frac{\omega(\Lambda^*[\alpha, \beta; \psi(\omega)])''}{(\Lambda^*[\alpha, \beta; \psi(\omega)])'} + 1\right\} > 0, \ \omega \in \mathbb{U}.$$

Since the conditions of the well-known Miller- Mocanu lemma (see [3, Theorem 3.6.1.]) are satisfied it follows that (4.8) implies $p(\omega) \prec \Lambda^*[\alpha, \beta; \psi(\omega)]$, and $\Lambda^*[\alpha, \beta; \psi(\omega)]$ is the best dominant of p, which prove our conclusions. □

Remark 4.2. Several special cases of Theorem 4.2 and Theorem 4.3 can be obtained by assigning some fixed values to the parameter involved in it.

5. Conclusion

We have obtained the interesting coefficient bounds involving analytic functions with respect to (η, τ)-symmetric points. Indeed, very few researchers have attempted the coefficient problems pertaining to analytic functions with respect to (η, τ)-symmetric points, as it is computationally tedious. Further, most of the studies in this area by various other authors involved the differential characterization subordinate to a Carathéodory function. But in this study, we have obtained interesting subordination conditions, inclusion and integral representation of the functions defined for a class of non-Carathéodory function.

Assertion of the Lemma 2.1 is true only if the superordinate function in (2.1) is convex, so the results that we obtained in Section 2 cannot be applied to functions that are subordinate to non-convex functions. Hence, there is a need to develop some tools or methods to obtain the coefficients for the functions subordinate to non-convex functions. In addition, we note that the impact of $\Lambda^*[\alpha, \beta; \psi(\omega)]$ is not the same in all conic regions. So, the following question arises: Are there any specific specialized regions in which the impact of $\Lambda^*[\alpha, \beta; \psi(\omega)]$ will be the same?

The study should be interesting when the ordinary derivative in Definition 1.1 is replaced with a multiplicative derivative. However, the presence of second order derivative in (1.12) will make such a study very complicated. Further, this study can be extended by replacing $p(\omega)$ in (1.4) with a Legendre polynomial, q-Hermite polynomial, Fibonacci sequence, or Chebyshev polynomial.

Acknowledgments

Authors would like to thank the referees and the academic editor for their comments and suggestions which helped us remove the mistakes. We also sincerely express our gratitude to the referees for their comments which led to improvements in the readability of the paper.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Funding

This research study received no external.

Data availability statement

No data was used to support this study.

Statement on author's contribution

All three authors contributed equally to this work. All the authors have read and agreed to the published version of the manuscript.

References

Al Sarari FSM, Frasin BA, Al-Hawary T, Latha S. 2016. A few results on generalized Janowski type functions associated with (jk)-symmetrical functions. Acta Univ Sapientiae Math. 8(2):195–205. doi: 10.1515/ausm-2016-0012.

Araci S, Karthikeyan KR, Murugusundaramoorthy G, Khan B. 2023. Φ-like analytic functions associated with a vertical domain. Math Inequal Appl. 26(4):935–949. doi: 10.7153/mia-2023-26-57.

Bulboacă T. 2005. Differential subordinations and superordinations. Recent result. Cluj-Napoca: House of Science Book Publisher.

Dziok J, Raina RK, Sokół J. 2011a. Certain results for a class of convex functions related to a shell-like curve connected with fibonacci numbers. Comput Math Appl. 61(9):2605–2613. doi: 10.1016/j.camwa.2011.03.006.

Dziok J, Raina RK, Sokół J. 2011b. On α-convex functions related to shell-like functions connected with Fibonacci numbers. Appl Math Comput. 218(3):996–1002. doi: 10.1016/j.amc.2011.01.059.

Dziok J, Raina RK, Sokół J. 2013. On a class of starlike functions related to a shell-like curve connected with Fibonacci numbers. Math Comput Modelling. 57(5–6):1203–1211. doi: 10.1016/j.mcm.2012.10.023.

Gandhi S. 2020. Radius estimates for three leaf function and convex combination of starlike functions. In: Deo N, Gupta V, Acu A, Agrawal P, editors. *Mathematical analysis. I. Approximation theory*. ICRAPAM 2018. Springer Proceedings in Mathematics & Statistics, Vols. 306. Singapore: Springer. doi: 10.1007/978-981-15-1153-0_15.

Hallenbeck DJ, Ruscheweyh S. 1975. Subordination by convex functions. Proc Am Math Soc. 52 (1):191–195. doi: 10.1090/S0002-9939-1975-0374403-3.

Ibrahim RW. 2020. On a Janowski formula based on a generalized differential operator, Commun. Fac Sci Univ Ank Ser A1 Math Stat. 69(2):1320–1328.

Ibrahim RW, Darus M. 2019. Subordination inequalities of a new salagean-difference operator. Int J Math Comput Sci. 14(3):573–582.
Karthikeyan KR. 2013. Some classes of analytic functions with respect to (j, k)- symmetric points. ROMAI J Math. 9(1):51–60.
Karthikeyan KR, Cho NE, Murugusundaramoorthy G. 2023. On classes of non-carathéodory functions associated with a family of functions starlike in the direction of the real axis. Axioms. 12(1):24. doi: 10.3390/axioms12010024.
Karthikeyan KR, Murugusundaramoorthy G, Nistor-Şerban A, Răducanu D. 2020. Coefficient estimates for certain subclasses of starlike functions of complex order associated with a hyperbolic domain. Bull Transilv Univ Braşov Ser III. 13((62)(2)):595–610. doi: 10.31926/but.mif.2020.13.62.2.17.
Karthikeyan KR, Murugusundaramoorthy G, Purohit SD, Suthar DL. 2021. Certain class of analytic functions with respect to symmetric points defined by q-calculus. J Math. 2021:1–9. Art. ID 8298848. doi: 10.1155/2021/8298848.
Khan MG, Ahmad B, Murugusundaramoorthy G, Mashwani WK, Yalçin S, Shaba TG, Salleh Z. 2022. Third Hankel determinant and Zalcman functional for a class of starlike functions with respect to symmetric points related with sine function. J Math Comput Sci. 25(1):29–36. doi: 10.22436/jmcs.025.01.04.
Kwon O, Sim Y. 2013. A certain subclass of Janowski type functions associated with k-symmetric points. Commun Korean Math Soc. 28(1):143–154. doi: 10.4134/CKMS.2013.28.1.143.
Libera RL. 1967. Univalent α-spiral functions. Canadian J Math. 19:449–456. doi: 10.4153/CJM-1967-038-0.
Liczberski P, Połubiński J. 1995. On (J, K)-Symmetrical Functions, Math Bohem. 120(1):13–28. doi:10.21136/MB.1995.125897
Ma WC and Minda DA unified treatment of some special classes of univalent functions, in *Proceedings of the Conference on Complex Analysis (Tianjin, 1992)*, 157–169, Conf. Proc. Lecture Notes Anal. I, Int. Press, Cambridge, MA.
Mendiratta R, Nagpal S, Ravichandran V. 2014. A subclass of starlike functions associated with left-half of the lemniscate of Bernoulli. Internat J Math. 25(9, 1450090):17. doi: 10.1142/S0129167X14500906.
Murat C, Karthikeyan KR, Murugusundaramoorthy G. 2023. Inequalities on a class of analytic functions defined by generalized Mittag-Leffler function. Filomat. 37(19):6277–6288.
Mustafa N, Korkmaz S. 2022. Coefficient bound estimates and szegö problem for a certain subclass of analytic and bi-univalent functions. Turkish J Sci. 7(3):211–218.
Mustafa N, Murugusundaramoorthy G. 2021. Second Hankel determinant for Mocanu type bi-starlike functions related to shell-shaped region. Turkish J Math. 45(3):1270–1286. doi: 10.3906/mat-2101-97.
Raina RK, Sokół J. 2015. On coefficient estimates for a certain class of starlike functions. Haceppt J Math Stat. 44(6):1427–1433. doi: 10.15672/HJMS.2015449676.
Raina RK, Sokół J. 2016. Fekete-szegö problem for some starlike functions related to shell-like curves. Math Slovaca. 66(1):135–140. doi: 10.1515/ms-2015-0123.
Raina RK, Sokół J. 2019. On a class of analytic functions governed by subordination. Acta univ Sapientiae Math. 11(1):144–155. doi: 10.2478/ausm-2019-0012.
Rogosinski W. 1943. On the coefficients of subordinate functions. Proc London Math Soc. 48 (2):48–82. doi: 10.1112/plms/s2-48.1.48.
Srivastava HM. 1968. On an extension of the Mittag-Leffler function. Yokohama Math J. 16:77–88.
Srivastava HM. 2021a. An introductory overview of fractional-calculus operators based upon the Fox-Wright and related higher transcendental functions. J Adv Eng Comput. 5(3):135–166. doi: 10.55579/jaec.202153.340.
Srivastava HM. 2021b. Some parametric and argument variations of the operators of fractional calculus and related special functions and integral transformations. J Nonlinear Convex Anal. 22:1501–1520.

Srivastava HM. 2021c. A survey of some recent developments on higher transcendental functions of analytic number theory and applied mathematics. Symmetry. 13(12):2294. doi: 10.3390/sym13122294.

Srivastava HM, Ahmad QZ, Khan N, Khan N, Khan B. 2019. Hankel and toeplitz determinants for a subclass of q-starlike functions associated with a general conic domain. Mathematics. 7 (181):1–15. Article ID. doi: 10.3390/math7020181.

Srivastava HM, Bansal M, Harjule P. 2018. A study of fractional integral operators involving a certain generalized multi-index Mittag-Leffler function. Math Methods Appl Sci. 41 (16):6108–6121. doi: 10.1002/mma.5122.

Srivastava HM, Bukhari SZH, Nazir M. 2018. A subclass of α-convex functions with respect to $(2j, k)$-symmetric conjugate points, Bull. Iranian Math Soc. 44(5):1227–1242. doi: 10.1007/s41980-018-0086-x.

Srivastava HM, Fernandez A, Baleanu D. 2019. Some new fractional-calculus connections between Mittag–Leffler functions. Mathematics. 7(6):485. doi: 10.3390/math7060485.

Srivastava HM, Khan B, Khan N, Ahmad QZ. 2019. Coefficient ine*q*ualities for $*q*$-starlike functions associated with the Janowski functions. Hokkaido Math J. 48(2):407–425. doi: 10.14492/hokmj/1562810517.

Srivastava HM, Kumar A, Das S, Mehrez K. 2022. Geometric properties of a certain class of Mittag–Leffler-type functions. Fractal Fract. 6(2):54. doi: 10.3390/fractalfract6020054.

Srivastava HM, Raza N, AbuJarad ESA, Srivastava G, AbuJarad MH. 2019. Fekete-Szegö inequality for classes of (p, q)-Starlike and (p, q)-convex functions. Rev Real Acad Cienc Exactas Fís Natur Ser A Mat (RACSAM). 113(4):3563–3584. doi: 10.1007/s13398-019-00713-5.

Srivastava HM, Tahir M, Khan B, Ahmad QZ, Khan N. 2019. Some general classes of q-starlike functions associated with the Janowski functions. Symmetry. 11(2):1–14. doi: 10.3390/sym11020292.

Srivastava HM, Tomovski Z. 2009. Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel, Appl App lMath Comput. 211(1):198–210. doi: 10.1016/j.amc.2009.01.055.