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Article

On the Ratio-Type Family of Copulas

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Abstract: Investigating dependence structures across various fields holds paramount importance. Consequently, the creation of new copula families plays a crucial role in developing more flexible stochastic models that address the limitations of traditional and sometimes impractical assumptions. The present article derives some reasonable conditions for validating a copula of the ratio-type form $uv/(1-\theta f(u)g(v))$. It includes numerous examples and discusses the admissible range of parameter θ, showcasing the diversity of copulas generated through this framework, such as Archimedean, non-Archimedean, positive dependent, and negative dependent copulas. The exploration extends to the upper bound of a general family of copulas, $uv/(1-\theta\phi(u,v))$, and important properties of the copula are discussed, including singularity, measures of association, tail dependence, and monotonicity. Furthermore, an extensive simulation study is presented, comparing the performance of three different estimators based on maximum likelihood, ρ-inversion, and the moment copula method.

Keywords: bivariate copula; ratio copula; Fréchet–Hoeffding limit; singularity; maximum likelihood; ρ-inversion; copula moments

MSC: 60E05; 62H05; 62H20

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1. Introduction

The copula industry has been thriving in the last decades, leading to the development of numerous methods to expand existing families or create new copulas. These efforts are motivated by the desire to introduce more flexible stochastic models that surpass the limitation of traditional and sometimes impractical assumptions about the distribution of multivariate random vectors. Various methods have been employed to introduce new parameters into or transformations of existing families of copulas, resulting in a versatile tool for understanding dependence among random variables (e.g., see [1–6]). For a comprehensive overview of historical developments, current findings, and future perspectives in this field, Durante and Sempi [7] and Hofert et al. [8] offer in-depth analyses incorporating the latest theories and insights.

The emergence of many copula families in recent times reflects the need to explore the structural dependencies across various domains such as finance, actuarial sciences, reliability engineering, life sciences, environmental sciences, hydrology, and survival analysis (e.g., see [9–14]).

The significance of copulas stems from the Sklar theorem [15], which asserts that for every random vector (X, Y) with a joint distribution function H and marginals F and G, there exists a copula C (uniquely determined when the random variables X and Y are continuous) linking the joint distribution function to F and G through the representation

$$H(x, y) = C(F(x), G(y)) \quad \text{for all } x, y \text{ in } \overline{\mathbb{R}}.$$

This statement divides the task of identifying a two-dimensional distribution function into two parts: identifying the marginal one-dimensional distribution functions F and G and selecting a suitable copula C that captures the dependence between the random variables. Hence, access to a diverse range of copulas is essential.

In the current paper, we examine ratio-type copulas of the form of

$$B_\theta(u,v) = \frac{uv}{1-\theta\phi(u,v)} \qquad 0 \le u,v \le 1, \tag{1}$$

where θ is a real-valued parameter and $\phi : [0,1] \times [0,1] \to \mathbb{R}$.

For a brief overview of works related to ratio-type copulas, one may refer to [1,3,16–19] and the citations therein. B_0 coincides with the independence copula, $\Pi(u,v) = uv$, whereas, for $\phi(u,v) = (1-u)(1-v)$ and $\theta \in [-1,1]$, B_θ corresponds to the Ali–Mikhail–Haq family of copula with parameter θ.

The mathematical complexity of Equation (1) makes a comprehensive study challenging and nearly unfeasible. Therefore, in Section 2, we definitely focus on scenario

$$D_\theta(u,v) = \frac{uv}{1-\theta f(u)g(v)} \qquad 0 \le u,v \le 1, \tag{2}$$

where $f, g : [0,1] \to \mathbb{R}$. Specifically, we examine the conditions on f and g under which D_θ is a valid copula and provide several examples along with the permissible range of parameter θ. Section 3 is devoted to the analysis of the upper bound of family B_θ including singularity, measures of association, tail dependence, and monotonicity. We conclude this section by estimating dependence parameter θ using three distinct methods: maximum likelihood, ρ-inversion, and the copula moment method. Finally, we provide concluding remarks and outline potential avenues for future research directions.
Throughout this paper, notations $\partial_u K$, $\partial_v K$, and $\partial_u \partial_v K$ stand for the respective partial derivatives and mixed partial derivative of function K with respect to u and v.

2. Generalized Family of Copulas

In this section, we study the family of functions defined in (2), where θ is a real-valued parameter and f and g are two non-zero differentiable functions defined over the unit interval.

Our objective is to identify the sufficient conditions for functions f and g so that D_θ is a valid copula. The following definition establishes the mathematical foundation of a bivariate copula within an absolutely continuous framework. We recall that a two-dimensional copula is function $C : [0,1]^2 \to [0,1]$ satisfying the following properties:

1. For every u, v in [0, 1],
$$C(u,0) = C(0,v) = 0,$$
and
$$C(u,1) = u \text{ and } C(1,v) = v.$$
2. For every u, v in [0, 1],
$$\partial_u \partial_v C(u,v) \ge 0,$$
where the mixed partial derivative is supposed to exist almost everywhere.

In the sequel, we proceed under the following assumptions.

Assumption 1.

1. $f(1) = g(1) = 0$.
2. *f and g are strictly monotone functions.*
3. $\dfrac{f(u)g(v)}{f(0)g(0)} \le 1 - uv$ *for all u and v in* $[0,1]$.

Remark 1.

1. *It is readily seen that* $D_\theta(u,1) = u$ *and* $D_\theta(1,v) = v$ *by the virtue of Assumption 1.1.*
2. *Assumptions 1.1 and 1.2 imply that, for* f *and* g *sharing the same monotonicity,* $f(u)g(v) \geq 0$, $f'(u)g'(v) > 0$ *and* $(f(u) - uf'(u))(g(v) - vg'(v)) \geq 0$, *for all* u *and* v *in* $[0,1]$. *Contrarily, if* f *and* g *do not have the same monotonicity, all the preceding expressions become nonpositive.*

Proposition 1. *We assume that Assumption 1 hold. Then,*

$$0 \leq D_\theta(u,v) \leq 1 \quad \forall (u,v) \in [0,1]^2 \quad \Longleftrightarrow \quad \begin{cases} \theta \leq \dfrac{1}{f(0)g(0)} & \text{for} \quad fg \geq 0 \\[2ex] \dfrac{1}{f(0)g(0)} \leq \theta & \text{for} \quad fg \leq 0 \end{cases} \qquad (\star)$$

Proof. In order for $D_\theta(u,v) \geq 0$ for all u, v in $[0,1]$, it is easy to show that θ must satisfy $(\star)$. Combining now $(\star)$ and Assumption 1.3 leads to $0 \leq D_\theta(u,v) \leq 1$, thereby concluding the proof of Proposition 1. □

We now wish to prove a sufficient condition for D_θ to be a two-dimensional copula. To that end, we make use of the methodology used in [20]. First, we establish a formula for mixed partial derivative $\partial_u \partial_v D_\theta$ via copula D_θ. An elementary calculation shows that

$$\partial_u D_\theta(u,v) = \frac{v}{1 - \theta f(u)g(v)} + \frac{\theta uvf'(u)g(v)}{(1 - \theta f(u)g(v))^2}.$$

Hence,

$$\begin{aligned} \partial_v \partial_u D_\theta(u,v) &= \frac{1 - \theta f(u)g(v) + \theta vf(u)g'(v)}{(1 - \theta f(u)g(v))^2} + \frac{(1 - \theta f(u)g(v))^2(\theta uf'(u)g(v) + \theta uvf'(u)g'(v))}{(1 - \theta f(u)g(v))^4} \\ &\quad + \frac{2\theta^2 uv(1 - \theta f(u)g(v))f(u)g(v)f'(u)g'(v)}{(1 - \theta f(u)g(v))^4} \\ &= \frac{1 - \theta f(u)g(v) + \theta vf(u)g'(v) + \theta uf'(u)g(v) + \theta uvf'(u)g'(v)}{(1 - \theta f(u)g(v))^2} + \frac{2\theta^2 uvf(u)g(v)f'(u)g'(v)}{(1 - \theta f(u)g(v))^3} \\ &= \frac{1 - \theta[(f(u) - uf'(u))(g(v) - vg'(v)) - 2D_\theta(u,v)f'(u)g'(v)]}{(1 - \theta f(u)g(v))^2}. \end{aligned}$$

We define, for the functions f and g,

$$\begin{aligned} \alpha_1 &= \min_{0 \leq u,v \leq 1} [(f(u) - uf'(u))(g(v) - vg'(v)) - 2uvf'(u)g'(v)], \\ \alpha_2 &= \max_{0 \leq u,v \leq 1} [(f(u) - uf'(u))(g(v) - vg'(v)) - 2uvf'(u)g'(v)]. \end{aligned}$$

We let $f'_-(1)$ and $g'_-(1)$ be the left derivatives of f and g at 1, respectively. We observe that $\alpha_1 \leq \min(f(0)g(0), -f'_-(1)g'_-(1))$ and $\alpha_2 \geq \max(f(0)g(0), -f'_-(1)g'_-(1))$, which implies that $\alpha_1 < 0 < \alpha_2$.

Theorem 1. *We assume that* f *and* g *satisfy Assumption 1. Then,* D_θ *is a valid copula provided that* $1/\alpha_1 \leq \theta \leq 1/\alpha_2$.

Proof. It is onerous but straightforward to show that $D_\theta(u,0) = D_\theta(0,v) = 0$ for the admissible range of θ, $[1/\alpha_1, 1/\alpha_2]$. We observe that if $1/\alpha_1 \leq \theta \leq 1/\alpha_2$, as per Proposition 1, $0 \leq D_\theta(u,v) \leq 1$ for all $(u,v) \in [0,1]^2$.

It will be shown later that

$$\theta D_\theta(u,v) f'(u) g'(v) \geq \theta uv f'(u) g'(v) \quad \forall (u,v) \in [0,1]^2. \tag{3}$$

Let us now prove that

$$\min_{0 \leq u,v \leq 1} \{1 - \theta[(f(u) - uf'(u))(g(v) - vg'(v)) - 2D_\theta(u,v) f'(u) g'(v)]\} \geq 0.$$

There are two cases to consider depending on the sign of θ. First, for $\theta \geq 0$, we have

$$\begin{aligned} &\min_{0 \leq u,v \leq 1} \{1 - \theta[(f(u) - uf'(u))(g(v) - vg'(v)) - 2D_\theta(u,v) f'(u) g'(v)]\} \\ \geq &\min_{0 \leq u,v \leq 1} \{1 - \theta[(f(u) - uf'(u))(g(v) - vg'(v)) - 2uvf'(u) g'(v)]\} \\ = &\ 1 - \theta\alpha_2. \end{aligned}$$

On the other hand, we observe that for $\theta \leq 0$,

$$\begin{aligned} &\min_{0 \leq u,v \leq 1} \{1 - \theta[(f(u) - uf'(u))(g(v) - vg'(v)) - 2D_\theta(u,v) f'(u) g'(v)]\} \\ \geq &\min_{0 \leq u,v \leq 1} \{1 - \theta[(f(u) - uf'(u))(g(v) - vg'(v)) - 2uvf'(u) g'(v)]\} \\ = &\ 1 - \theta\alpha_1. \end{aligned}$$

To complete the proof of the theorem, we verify Inequality (3). We note that $D_\theta(u,v) \leq uv$ for all u and v satisfying $\theta f(u) g(v) \leq 0$. Further, and since $D_\theta(u,v) \geq 0$, it is easily seen that $D_\theta(u,v) \geq uv$ for all u and v satisfying $\theta f(u) g(v) \geq 0$. By Remark 1.2, fg and $f'g'$ share the same sign. This completes the proof of Inequality (3) and that of Theorem 1. □

Let us now revisit some fundamental definitions regarding the concordance ordering of copulas, quadrant dependence, and tail monotonicity (e.g., see [21,22]).

Definition 1. *We let C_1 and C_2 be two copulas. We say that C_2 is more concordant than C_1, denoted $C_1 \prec C_2$, if $C_1(u,v) \leq C_2(u,v)$ for all $0 \leq u,v \leq 1$.*

Definition 2. *A family of copulas, C_α, is positively ordered if $C_{\alpha_1} \prec C_{\alpha_2}$ whenever $\alpha_1 \leq \alpha_2$, and negatively ordered if $C_{\alpha_1} \succ C_{\alpha_2}$ whenever $\alpha_1 \leq \alpha_2$.*

For any $(u,v) \in [0,1]^2$, we have

$$\partial_\theta D_\theta(u,v) = \frac{uvf(u)g(v)}{(1 - \theta f(u) g(v))^2}.$$

If f and g exhibit identical monotonicity, then D_θ is positively ordered with respect to θ.

Definition 3. *A family of copulas, C_α, is positively quadrant dependent (PQD) if $C_\alpha \succ \Pi$. Negative quadrant dependence (NQD) is defined analogously by reversing the sense of the concordance ordering.*

Since $D_0 = \Pi$, copula D_θ is PQD (NQD) if $\theta f(u) g(v) \geq 0$ ($\theta f(u) g(v) \leq 0$) for all $0 \leq u,v \leq 1$.

Definition 4. *We let (X,Y) be a pair of continuous random variables whose copula is C. Then, Y is said to be left tail decreasing in X [LTD($Y|X$)] if and only if for any $0 \leq v \leq 1$, $C(u,v)/u$ is a nonincreasing function of u.*

Proposition 2. *We let (X,Y) be a continuous random pair with copula D_θ. Then, both LTD($Y|X$) and LTD($X|Y$) are in force if and only if one of the following conditions hold:*

1. *f and g share the same monotonicity and θ is positive.*
2. *f and g do not share the same monotonicity and θ is negative.*

Proof. For $0 \leq v \leq 1$, we obtain

$$\frac{\partial C(u,v)}{\partial u} - \frac{C(u,v)}{u} = \frac{\theta u v f' g}{(1-\theta f(u) g(v))^2}.$$

Also, for $0 \leq u \leq 1$, we have

$$\frac{\partial C(u,v)}{\partial v} - \frac{C(u,v)}{v} = \frac{\theta u v f g'}{(1-\theta f(u) g(v))^2}.$$

The proof of the proposition concludes by noting that $f'g$ and fg' have the same sign, along with Corollary 5.2.6 of [22]. □

Let us now investigate another concept of dependence known as tail dependence. As pointed out in [21,22], the lower/upper tail dependence coefficients for copula D_θ are given by

$$\lambda_L = \lim_{u \to 0^+} \frac{D_\theta(u,u)}{u} = \lim_{u \to 0^+} \frac{u}{1-\theta f(u) g(u)} = \begin{cases} -\frac{(fg)(0)}{(fg)'(0)} & \text{if } \theta = \frac{1}{(fg)(0)} \\ 0 & \text{elsewhere,} \end{cases}$$

and

$$\lambda_U = \lim_{u \to 1^-} \frac{1-2u+D_\theta(u,u)}{1-u} = \lim_{u \to 1^-} 1 + \frac{u^2 - u(1-\theta f(u) g(u))}{(1-u)(1-\theta f(u) g(u))} = 0.$$

To conclude, D_θ exhibits lower tail dependence for $\theta = 1/(fg)(0)$.

Example 1. *In the following, we present some examples of copulas in accordance to Equation (2) and Assumption 1.*

To conclude this section, we illustrate the range of values of well-known measures of association for the copulas listed in Table 1 through numerical methods.

Table 1. Example of ratio-type copulas based on the new construction.

	$f(u)$	$g(u)$	**Parameters**	θ
$D_\theta^{(1)}$	$1-u$	$1-v$	$\alpha_1 = -1, \alpha_2 = 1$	$[-1, 1]$
$D_\theta^{(2)}$	$\ln(2-u)$	$\ln(2-v)$	$\alpha_1 = -1, \alpha_2 = \ln(2)$	$\left[-1, \frac{1}{\ln(2)}\right]$
$D_\theta^{(3)}$	$-\frac{2}{\pi}\cos\left(\frac{\pi u}{2}\right)$	$-\frac{2}{\pi}\cos\left(\frac{\pi v}{2}\right)$	$\alpha_1 = -1, \alpha_2 = \frac{2}{\pi}$	$\left[-1, \frac{\pi}{2}\right]$
$D_\theta^{(4)}$	$1-u$	$-\ln(2-v)$	$\alpha_1 = -1, \alpha_2 = 1$	$[-1, 1]$
$D_\theta^{(5)}$	$-\frac{2}{\pi}\cos\left(\frac{\pi u}{2}\right)$	$1-v$	$\alpha_1 = -1, \alpha_2 = 1$	$[-1, 1]$
$D_\theta^{(6)}$	$\frac{\ln(2-u)}{\ln(2)}$	$-\frac{2}{\pi}\cos\left(\frac{\pi v}{2}\right)$	$\alpha_1 = -1, \alpha_2 = \frac{1}{\ln(2)}$	$[-1, \ln(2)]$
$D_\theta^{(7)}$	$(1-u)e^{-u}$	$(1-v)e^{-v}$	$\alpha_1 = -e^{-2}, \alpha_2 = 1$	$[-e^2, 1]$
$D_\theta^{(8)}$	$1-u^{\frac{3}{2}}$	$1-v^{\frac{3}{2}}$	$\alpha_1 = -2.25, \alpha_2 = 1.5$	$[-0.44, 0.67]$

The generalized family of copulas described in (2) contains a wide range of copulas, including symmetric and non-symmetric ones. Notably, all copulas listed in Table 1 except for the Ali–Mikhail–Haq copula are non-Archimedean. Additionally, this generalized family of copulas accommodates both negative and positive dependence by expanding the scope of association measures related to the AMH copula. For example, as shown in Table 2, we observe that $\rho_{min}(D_\theta^{(7)})$ is less than $\rho_{min}(D_\theta^{(1)})$, while $\rho_{max}(D_\theta^{(3)})$ exceeds $\rho_{max}(D_\theta^{(1)})$. However, the family of copulas D_θ is not comprehensive and lacks completeness in the sense that it does not cover the Fréchet–Hoeffding lower and upper bounds.

Table 2. Numerical analysis of Spearman's ρ, Kendall's τ, Gini's γ, and Blomqvist's β.

	ρ_θ	τ_θ	γ_θ	β_θ
$D_\theta^{(1)}$	$[-0.2711, 0.4784]$	$[-0.1817, 0.3333]$	$[-0.0540, 0.0955]$	$[-0.2000, 0.3333]$
$D_\theta^{(2)}$	$[-0.1956, 0.4191]$	$[-0.1307, 0.2834]$	$[-0.0387, 0.0837]$	$[-0.1412, 0.3109]$
$D_\theta^{(3)}$	$[-0.2243, 0.5801]$	$[-0.1500, 0.3964]$	$[-0.0448, 0.1188]$	$[-0.1685, 0.4669]$
$D_\theta^{(4)}$	$[-0.2311, 0.3457]$	$[-0.1546, 0.332]$	$[-0.0458, 0.0689]$	$[-0.1686, 0.2543]$
$D_\theta^{(5)}$	$[-0.2470, 0.3811]$	$[-0.1654, 0.2575]$	$[-0.0493, 0.0766]$	$[-0.1837, 0.2905]$
$D_\theta^{(6)}$	$[-0.2096, 0.4630]$	$[-0.1401, 0.3138]$	$[-0.0417, 0.0935]$	$[-0.1544, 0.3575]$
$D_\theta^{(7)}$	$[-0.5311, 0.1578]$	$[-0.3666, 0.1069]$	$[-0.1072, 0.0310]$	$[-0.4046, 0.1013]$
$D_\theta^{(8)}$	$[-0.2124, 0.5025]$	$[-0.1420, 0.3418]$	$[-0.0423, 0.1015]$	$[-0.1566, 0.3862]$

3. Upper Bound of Family B_θ

The following section is consecrated on investigating the properties of a new copula, the upper bound of family B_θ defined in (1). We begin by establishing the acceptable range of θ to ensure a valid copula and then derive its corresponding absolutely continuous and singular components. Subsequently, we address the concordance measures of the novel copula, including Spearman's ρ, Kendall's τ, Gini's γ, and Blomqvist's β, and present them in closed forms. This is then followed by a brief investigation of monotonicity and tail dependency properties. Finally, we conclude by estimating dependence parameter θ with three different methods: maximum likelihood, ρ-inversion, and the copula moment approach.

Remark 2. *It is easily verified that the family $\{B_\theta, \theta \in [-1, 1]\}$, defined in (1), is positively ordered for nonnegative ϕ and negatively ordered for nonpositive ϕ since*

$$\partial_\theta B_\theta(u, v) = \frac{uv\phi(u, v)}{(1 - \theta\phi(u, v))^2}.$$

We let B_θ be a member of the family expressed in (1). Since $B_0 = \Pi$, we remark that B_θ is PQD for $\theta \geq 0$ and NQD for $\theta \leq 0$, provided that ϕ is a nonnegative function. In the contrary case, for nonpositive ϕ, B_θ is NQD for $\theta \geq 0$ and PQD for $\theta \leq 0$.

3.1. Upper Bound Copula

In the following, we show that the family of copulas B_θ includes the Fréchet–Hoeffding upper bound, $M(u, v) = \min(u, v)$. It is readily checked that the upper bound of family B_θ is reached when $(\theta = 1, \phi \geq 0)$ or $(\theta = -1, \phi \leq 0)$. We write

$$\frac{uv}{1 - \phi_1(u, v)} = \min(u, v) \iff \phi_1(u, v) = \frac{\min(u, v) - uv}{\min(u, v)} = \min(1 - u, 1 - v),$$

or

$$\frac{uv}{1 + \phi_2(u, v)} = \min(u, v) \iff \phi_2(u, v) = \frac{uv - \min(u, v)}{\min(u, v)} = -\min(1 - u, 1 - v).$$

Proposition 3. *The bivariate function*

$$C_\theta(u,v) = \frac{uv}{1-|\theta|\min(1-u,1-v)}, \quad (u,v) \in [0,1]^2$$

is a copula if and only if $|\theta| \leq 1$.

Copula C_θ has been previously introduced in the literature, particularly derived from the family of symmetric bivariate copulas studied in [23],

$$C_h(u,v) = \min(u,v)h(\max(u,v)),$$

with a generator $h(x) = x/(1-|\theta|(1-x))$ in our case. Furthermore, Example 3.2 in [24] refers to C_θ, $0 \leq \theta \leq 1$, as a generalized Ali–Mikhail–Haq copula. Making use now of Theorem 2.1 of the same reference [23], we establish that C_θ is a copula if and only if $|\theta| \leq 1$.

It is also worth mentioning that the copula, C_θ, can be written as

$$C_\theta(u,v) = \frac{\Pi(u,v)M(u,v)}{F(u,v)},$$

where $M(u,v) = \min(u,v)$ is the Fréchet–Hoeffding upper bound and $F(u,v) = (1-|\theta|)M(u,v) + |\theta|\Pi(u,v)$ is a member of the Fréchet copula family.

Without loss of generality, we focus our discussion on the properties of C_θ for positive θ.

3.2. Simulation from the Copula C_θ

The following presents an algorithm for simulating data from copula C_θ using the inverse method. To this end, we let (U,V) be a pair of uniform random variables with copula D_θ. Note that for fixed $u \in [0,1]$,

$$c_u(v) = \partial_u C(u,v) = \begin{cases} \frac{v(1-\theta)}{(1-\theta+\theta u)^2} & \text{if} \quad v < u, \\ \\ \frac{v}{1-\theta+\theta v} & \text{if} \quad v > u. \end{cases} \tag{4}$$

We define $a_0(u) = 0$, $a_1(u) = \dfrac{u(1-\theta)}{(1-\theta+\theta u)^2}$, $a_2(u) = \dfrac{u}{1-\theta+\theta u}$, $a_3(u) = 1$ and let $A_1 = \{t : a_0(u) \leq t < a_1(u)\}$, $A_2 = \{t : a_1(u) \leq t \leq a_2(u)\}$ and $A_3 = \{t : a_2(u) < t \leq a_3(u)\}$.

Note that function $v \mapsto c_u(v)$ is discontinuous in $v = u$. Thus, the generalized inverse of c_u is given by

$$\begin{aligned} c_u^{-1}(t) &= \inf\{v : c_u(v) \geq t\} \\ &= \frac{t(1-\theta+\theta u)^2}{1-\theta}\mathbf{I}_{A_1}(t) + u\,\mathbf{I}_{A_2}(t) + \frac{t(1-\theta)}{1-\theta t}\mathbf{I}_{A_3}(t), \end{aligned}$$

where $\mathbf{I}_A(\cdot)$ stands for the indicator function of A.

The algorithm below generates random numbers from copula C_θ:

- Generate uniform aleas u and t.
- Set $v = c_u^{-1}(t)$.
- The desired pair is (u,v).

If U and T are independent uniform [0,1] random variables as in the preceding algorithm, we remark that

$$\begin{aligned} P(U=V) = P(a_1(U) \leq T \leq a_2(U)) &= \int_0^1 P(a_1(u) \leq T \leq a_2(u))\, du \\ &= \int_0^1 \left[\frac{u}{1-\theta+\theta u} - \frac{u(1-\theta)}{(1-\theta+\theta u)^2} \right] du \\ &= \frac{2-\theta}{\theta} + \frac{2(1-\theta)}{\theta^2} \ln(1-\theta). \end{aligned} \tag{5}$$

Figure 1 showcases scatterplots depicting simulations of the proposed family of copulas C_θ. Each scatterplot comprises 100 pairs of points generated by the aforementioned algorithm, varying across different values of θ.

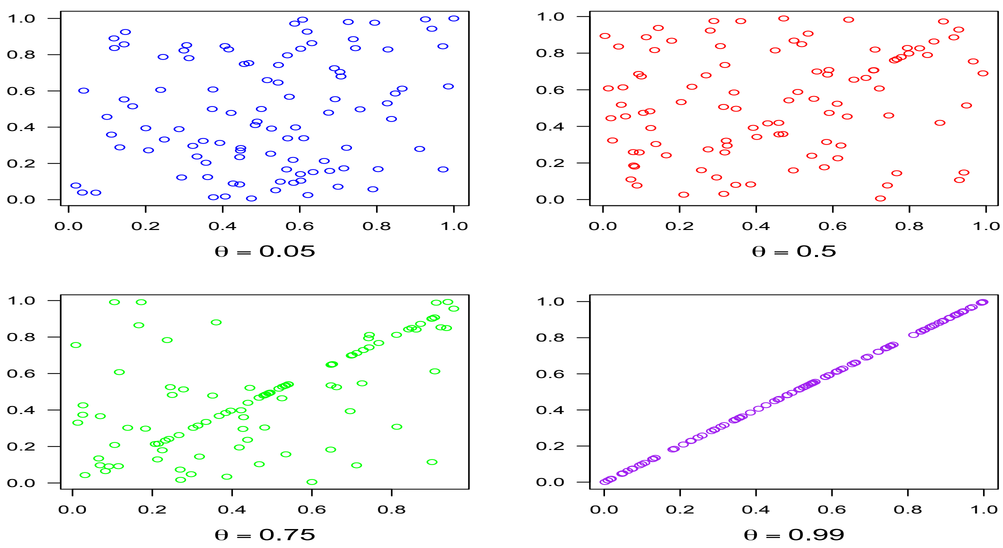

Figure 1. Scatterplots of 100 points from copula C_θ.

For illustrative purposes and visual validation, Figure 2 displays the plots of copula C_θ for different choices of parameter θ.

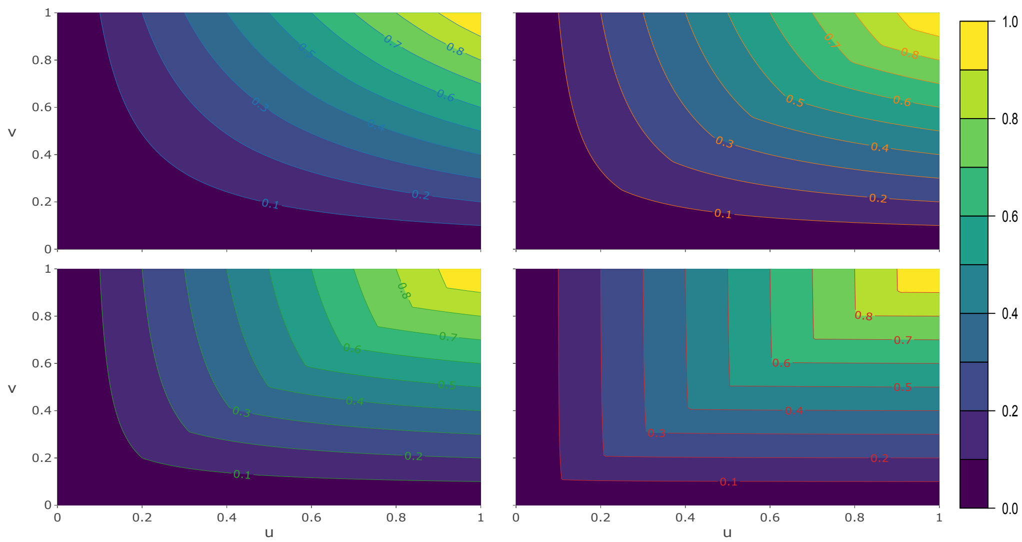

Figure 2. Intensity contours for C_θ where $\theta = 0.05$ (**upper left**), $\theta = 0.5$ (**upper right**), $\theta = 0.75$ (**lower left**) and $\theta = 0.99$ (**lower right**).

3.3. Singularity

It is important to note that the proposed copula is not absolutely continuous. It possesses a probability mass concentrated on line $u = v$ in $[0,1]^2$. The next result presents explicit forms for both continuous and singular components A_θ and S_θ, respectively.

Proposition 4. *For all $0 \leq u \neq v \leq 1$, the absolutely continuous and singular components of copula C_θ are defined by*

$$A_\theta(u,v) = \frac{2(1-\theta)}{\theta^2} \ln\left(\frac{1-\theta+\theta\min(u,v)}{1-\theta}\right) - \frac{(1-\theta)\min(u,v)}{\theta}\left(\frac{1}{1-\theta+\theta u} + \frac{1}{1-\theta+\theta v}\right),$$

and

$$S_\theta(u,v) = C_\theta(u,v) - A_\theta(u,v).$$

Proof. Standard calculations show that, for all $0 \leq x \neq y \leq 1$,

$$\partial_x \partial_y C_\theta(x,y) = \frac{1-\theta}{(1-\theta+\theta\max(x,y))^2} \tag{6}$$

Hence, the continuous part is calculated, for all $0 \leq u < v \leq 1$, by

$$\begin{aligned}
A_\theta(u,v) &= \int_0^u \int_0^v \partial_x \partial_y C_\theta(x,y)\, dy dx \\
&= \int_0^u \left[\int_0^x \frac{1-\theta}{(1-\theta+\theta\max(x,y))^2} dy + \int_x^v \frac{1-\theta}{(1-\theta+\theta\max(x,y))^2} dy\right] dx \\
&= (1-\theta)\int_0^u \left[\int_0^x \frac{1}{(1-\theta+\theta x)^2} dy + \int_x^v \frac{1}{(1-\theta+\theta y)^2} dy\right] dx \\
&= (1-\theta)\int_0^u \left[\frac{x}{(1-\theta+\theta x)^2} + \frac{1}{\theta}\left(\frac{1}{1-\theta+\theta x} - \frac{1}{1-\theta+\theta v}\right)\right] dx \\
&= \frac{2(1-\theta)}{\theta^2} \ln\left(\frac{1-\theta+\theta u}{1-\theta}\right) - \frac{(1-\theta)u}{\theta}\left(\frac{1}{1-\theta+\theta u} + \frac{1}{1-\theta+\theta v}\right).
\end{aligned}$$

Similarly, one obtains, for $0 \leq v < u \leq 1$,

$$A_\theta(u,v) = \frac{2(1-\theta)}{\theta^2} \ln\left(\frac{1-\theta+\theta v}{1-\theta}\right) - \frac{(1-\theta)v}{\theta}\left(\frac{1}{1-\theta+\theta u} + \frac{1}{1-\theta+\theta v}\right).$$

This completes the proof of the proposition. □

The C_θ-measure of the singular component of copula C_θ is given by

$$S_\theta(1,1) = 1 - A_\theta(1,1) = \frac{2-\theta}{\theta} + \frac{2(1-\theta)}{\theta^2} \ln(1-\theta).$$

As discussed earlier in (5), quantity $S_\theta(1,1)$ represents the probability of $[U = V]$ where (U,V) is the vector of uniform random variables whose distribution is C_θ. Furthermore, standard calculations show that $S_\theta(1,1)$ is an increasing function of θ such that

$$\lim_{\theta \to 0^+} S_\theta(1,1) = 0 \quad \text{and} \quad \lim_{\theta \to 1^-} S_\theta(1,1) = 1.$$

Proposition 5. *The copula density of C_θ is obtained by*

$$c_\theta(u,v) = \frac{1-\theta}{(1-\theta+\theta\max(u,v))^2} \mathbf{I}_{[u \neq v]} + \frac{\theta u^2}{(1-\theta+\theta u)^2} \mathbf{I}_{[u=v]}. \tag{7}$$

Proof. In light of Equation (4), we remark that conditional copula $C_\theta(\cdot|u)$ has a jump discontinuity at u totaling a mass of $a(u)$, where

$$a(u) = \lim_{v \to u^+} \partial_u C_\theta(u, v) - \lim_{v \to u^-} \partial_u C_\theta(u, v) = \frac{\theta u^2}{(1 - \theta + \theta u)^2}.$$

Making use of Equation (6) and Theorem 1.1 of [21] completes the proof of the proposition. □

To illustrate, copula density c_θ for $\theta = 0.05$, 0.5, 0.75 and 0.99 appears in Figure 3.

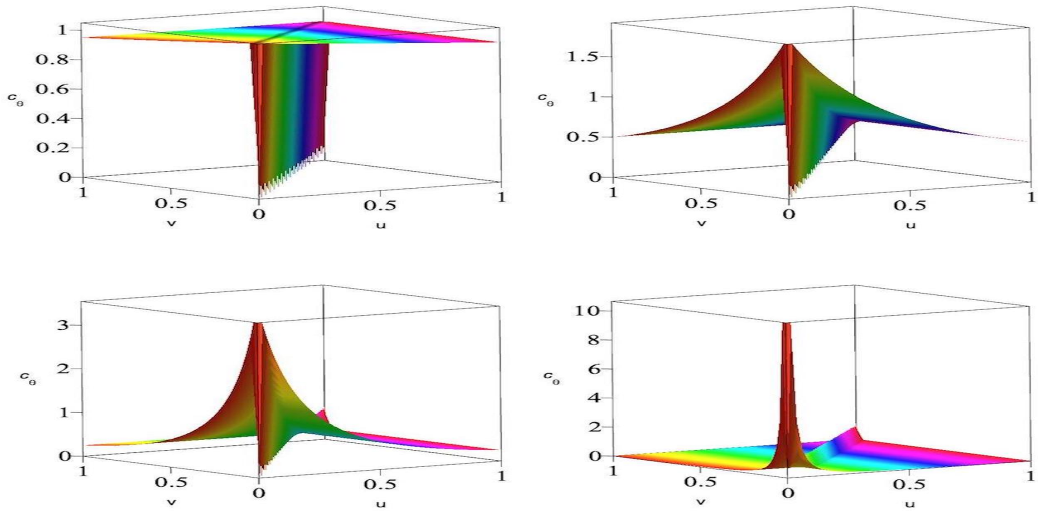

Figure 3. Copula density plots for $\theta = 0.05$ (**upper left**), $\theta = 0.5$ (**upper right**), $\theta = 0.75$ (**lower left**) and $\theta = 0.99$ (**lower right**).

3.4. Measures of Association

To explore the extent of positive dependence characterized by copula C_θ, we provide an overview of the commonly employed measures of association for bivariate copulas (see [22]).

Proposition 6. *We let C_θ be a member of the family of copulas defined in Proposition 3. Spearman's ρ, Kendall's τ, Gini's γ, and Blomqvist's β can be expressed as*

$$\begin{aligned}
\rho_\theta &= \frac{12(1-\theta)^3 \ln(1-\theta) - 3\theta^4 + 22\theta^3 - 30\theta^2 + 12\theta}{\theta^4}, \\
\tau_\theta &= \frac{-12(1-\theta)^2 \ln(1-\theta) - \theta^4 - 4\theta^3 + 18\theta^2 - 12\theta}{\theta^4}, \\
\gamma_\theta &= \frac{4(1-\theta)\left[2\ln\left(1 - \frac{\theta}{2}\right) - (1-\theta)\ln(1-\theta)\right] - 2\theta^3 + 3\theta^2}{\theta^3}, \\
\beta_\theta &= \frac{\theta}{2-\theta}.
\end{aligned}$$

Proof. The above expressions can be derived directly from Proposition 3.4 in [23], using the generator function $h(x) = x/(1 - \theta + \theta x)$. □

It is guaranteed by means of Remark 2 that the measures of dependence ρ_θ, τ_θ, γ_θ, and β_θ are nondecreasing functions with respect to dependence parameter θ. Moreover, the aforementioned measures satisfy the following properties, as depicted in Figure 4:

$$\begin{aligned}
&\lim_{\theta\to 0^+} \rho_\theta = \lim_{\theta\to 0^+} \tau_\theta = \lim_{\theta\to 0^+} \gamma_\theta = \lim_{\theta\to 0} \beta_\theta = 0, \\
&\lim_{\theta\to 1^-} \rho_\theta = \lim_{\theta\to 1^-} \tau_\theta = \lim_{\theta\to 1^-} \gamma_\theta = \lim_{\theta\to 1} \beta_\theta = 1, \\
&\tau_\theta < \beta_\theta < \gamma_\theta < \rho_\theta \quad \text{for} \quad 0 < \theta < \theta_0, \\
&\tau_\theta < \gamma_\theta < \beta_\theta < \rho_\theta \quad \text{for} \quad \theta_0 < \theta < 1,
\end{aligned}$$

where $\theta_0 = 0.6445832$ is the unique solution of equation $\gamma_\theta = \beta_\theta$. Furthermore, it should be noted, as illustrated in the figure below, that the measures of dependence γ_θ and β_θ almost coincide, while ρ_θ and τ_θ are linearly connected through the following equation:

$$\rho_\theta + (1-\theta)\tau_\theta = \theta.$$

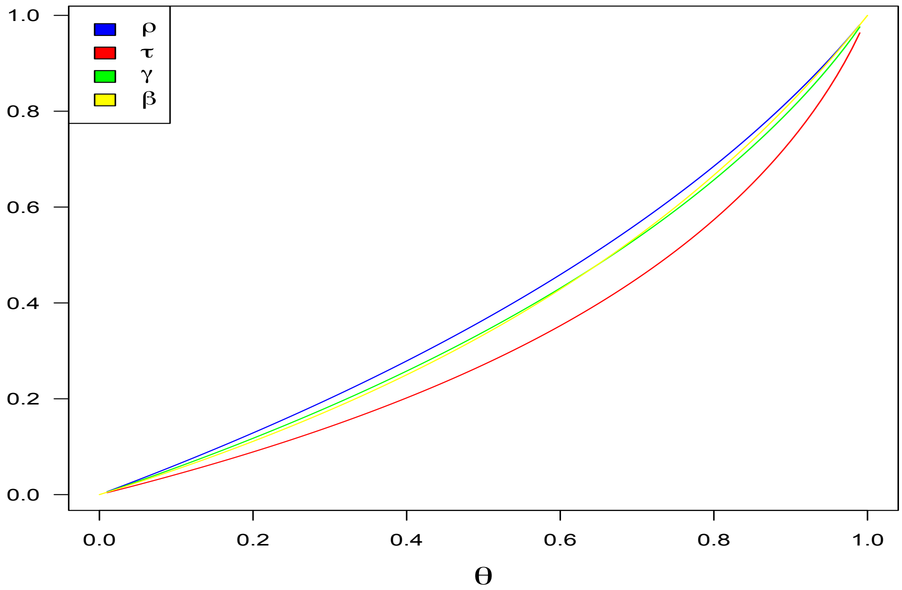

Figure 4. Graph of Spearman's ρ, Kendall's τ, Gini's γ, and Blomqvist's β.

3.5. Tail Dependence and Monotonicity

The concept of tail dependence relates to the amount of dependence in the upper-quadrant tail (lower-quadrant tail) of a bivariate distribution. It measures the probability of one variable being extreme given that other is extreme. In numerous financial contexts, tail dependence assumes pivotal importance when examining the impact of extremal events.

As a consequence of Proposition 3.3 from [23], we obtain

$$\lambda_L(C_\theta) = \begin{cases} 1 & \text{if } \theta = 1, \\ 0 & \text{elsewhere,} \end{cases} \quad \text{and} \quad \lambda_U(C_\theta) = \theta.$$

Consequently, C_θ is lower tail dependent for $\theta = 1$ and upper tail dependent for any $\theta \neq 0$.

Let us now recall some definitions concerning tail monotonicity, stochastic monotonicity, and corner set monotonicity (see [22] for a complete study).

Definition 5. *We let (X, Y) be a pair of random variables whose copula is D. Then,*

1. *Y is said to be right-tail increasing in X [RTI($Y|X$)] if and only if for any $0 \leq v \leq 1$, $[v - D(u,v)]/(1-u)$ is a nonincreasing function of u;*
2. *Y is stochastically increasing in X [SI($Y|X$)] if and only if for any $0 \leq v \leq 1$ and for all most u, $\partial D(u,v)/\partial u$ is a nonincreasing function of u.*

Definition 6.

1. *X and Y are left-corner set decreasing [LCSD(X, Y)] if* $P(X \leq x, Y \leq y | X \leq x', Y \leq y')$ *is nonincreasing in* x' *and* y' *for all* x *and* y.
2. *Function* f *defined from* $\overline{\mathbb{R}}^2$ *to* $\overline{\mathbb{R}}$ *is totally positive of order two [*TP_2*], if* $f(x, y) \geq 0$ *on* $\overline{\mathbb{R}}^2$ *and*

$$\begin{vmatrix} f(x,y) & f(x,y') \\ f(x',y) & f(x',y') \end{vmatrix} \geq 0,$$

for all $x \leq x'$ *and* $y \leq y'$.

As a direct result of Proposition 3.2 from [23], we deduce the following properties.

Proposition 7. *We let* (X, Y) *be a pair of continuous random variables whose copula is* C_θ*. Then,*

1. *Y is stochastically increasing in X;*
2. *Y is left-tail decreasing in X;*
3. *X and Y are left-corner set decreasing.*

Based on Corollary 5.2.17 of [22], it follows that C_θ is TP_2. It is also immediate from Proposition 7 that inequality $0 \leq \tau \leq \rho$ holds, as evidenced by Capéraà and Genest [25]. This finding aligns with our earlier observation.

3.6. Parameter Estimation via Maximum Likelihood

In the following, we address the problem of computing the maximum likelihood estimator, $\hat{\theta}_{\text{ML}}$, of the unknown parameter of dependence θ. To achieve this, we consider a bivariate random sample $\mathbf{w} := (\mathbf{w}_1, \ldots, \mathbf{w}_n)^t$ from copula C_θ with $\mathbf{w}_i = (u_i, v_i)$ for $i = 1, \ldots, n$, and we define $E \subset \{1, 2, \cdots, n\}$, the set of indexes of points in the sample lying on curve $\{(u, v) | u = v\}$.

From expression (7), we obtain the likelihood function for $\theta \in [0, 1]$

$$\begin{aligned} L(\mathbf{w}, \theta) &= \prod_{i \notin E} \frac{1-\theta}{(1-\theta+\theta \max(u_i, v_i))^2} \prod_{i \in E} \frac{\theta u_i^2}{(1-\theta+\theta u_i)^2} \\ &= \theta^m (1-\theta)^{n-m} \prod_{i=1}^{n} \frac{1}{(1-\theta+\theta \max(u_i, v_i))^2} \prod_{i \in E} u_i^2, \end{aligned}$$

where m is the cardinal of E.

Hence, $\hat{\theta}_{\text{ML}}$ can be obtained by maximizing the log-likelihood function, $\ell(\mathbf{w}, \theta)$, with respect to parameter θ:

$$\partial_\theta \ell(\mathbf{w}, \theta) = \frac{m - n\theta}{\theta(1-\theta)} + 2 \sum_{i=1}^{n} \frac{1 - \max(u_i, v_i)}{1 - \theta + \theta \max(u_i, v_i)}. \tag{8}$$

Clearly, the solution of the likelihood equation cannot be obtained in a simple closed form and numerical techniques are required consequently.

The maximum likelihood estimator $\hat{\theta}_{MLE}$ is asymptotically normal:

$$\sqrt{n}(\hat{\theta}_{\text{ML}} - \theta) \to \mathcal{N}(0, I^{-1}(\theta)) \quad \text{as } n \to \infty.$$

The fisher information, $I(\theta)$, can be written as

$$I(\theta) = -E\left[\partial_\theta^2 \ell(\mathbf{w}, \theta)\right], \tag{9}$$

where

$$\partial_\theta^2 \ell(\mathbf{w}, \theta) = \frac{-n\theta^2 + 2m\theta - m}{\theta^2(1-\theta)^2} + 2 \sum_{i=1}^{n} \frac{(1 - \max(u_i, v_i))^2}{(1 - \theta + \theta \max(u_i, v_i))^2}. \tag{10}$$

Making use again of Equation (7), we start by computing the following integral:

$$\begin{aligned}
&\iint_{[0,1]^2} \frac{(1-\max(u,v))^2}{(1-\theta+\theta\max(u,v))^2}\, c_\theta(u,v)\,dudv \\
&= (1-\theta)\iint_{[0,1]^2} \frac{(1-\max(u,v))^2}{(1-\theta+\theta\max(u,v))^4}\,dudv + \theta\int_0^1 \frac{u^2(1-u)^2}{(1-\theta+\theta u)^4}\,du \\
&= (1-\theta)\int_0^1\left[\int_0^u \frac{(1-u)^2}{(1-\theta+\theta u)^4}\,dv + \int_u^1 \frac{(1-v)^2}{(1-\theta+\theta v)^4}\,dv\right]du + \theta\int_0^1 \frac{u^2(1-u)^2}{(1-\theta+\theta u)^4}\,du \\
&= \int_0^1\left[\frac{(1-\theta)u(1-u)^2}{(1-\theta+\theta u)^4} + \frac{(1-\theta)(1-u)^3}{3(1-\theta+\theta u)^3} + \frac{\theta u^2(1-u)^2}{(1-\theta+\theta u)^4}\right]du \\
&= \frac{6(1-\theta)\ln(1-\theta)-\theta^3-3\theta^2+6\theta}{3\theta^4(1-\theta)}. \qquad (11)
\end{aligned}$$

Combining now (9), (10), and (11) leads to

$$I(\theta) = \frac{-12n(1-\theta)^2\ln(1-\theta)+n\theta^4-(6m+4n)\theta^3+(3m+18n)\theta^2-12n\theta}{3\theta^4(1-\theta)^2}.$$

3.7. Simulation Study

To evaluate the performance of the maximum likelihood estimator in small samples, we considered n mutually independent copies, $\mathbf{W}_1,\ldots,\mathbf{W}_n$, of the vector comprising unit uniform random variables U and V with associated copula C_θ. The estimator of dependence parameter θ was derived using routine function *optim* in the R 4.2.1 software. Various sample sizes were examined with 500 replications for each scenario.

Our results, as detailed in Table 3, encompass estimator $\hat{\theta}_{\mathrm{ML}}$, its bias, mean squared error (MSE), and a 95% asymptotic confidence interval for θ. Across the different scenarios investigated, simulations consistently demonstrated the efficacy of $\hat{\theta}_{\mathrm{ML}}$ as an estimator for dependence parameter θ. We observed that the performance of the estimator improved with larger n as the confidence intervals became narrower. Furthermore, the bias and MSE of $\hat{\theta}_{\mathrm{ML}}$ shrank with the number of observations n, indicating that the greater the number of observations, the more reliable the estimate. This trend was particularly evident when analyzing the behavior of the MSE, as outlined in Figure 5.

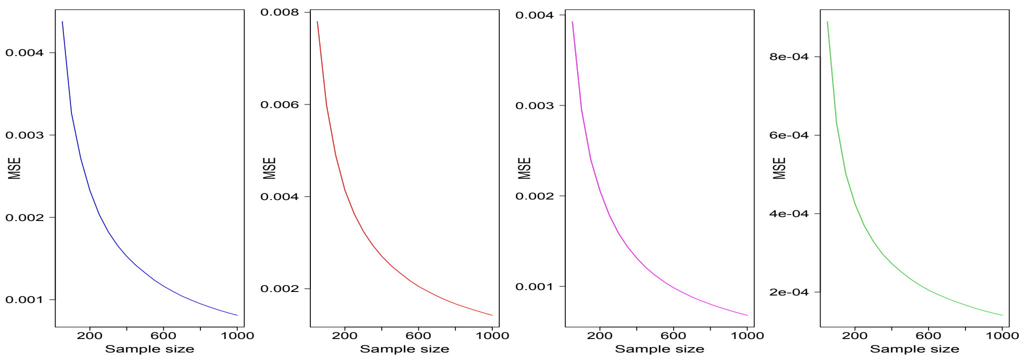

Figure 5. Mean Squared Error (MSE) of the Maximum Likelihood Estimator for $\theta = 0.1, 0.3, 0.7$ and 0.9 (ordered from left to right).

Following the classical method of moments approach, we considered two estimators for parameter of dependance θ, namely the ρ-inversion and the copula moment estimators.

Table 3. Maximum likelihood estimation for θ.

	n	$\hat{\theta}_{\mathrm{ML}}$	Bias	MSE	95% CI
	50	0.1019	0.0019	0.0044	(0,0.2363)
	100	0.0992	−0.0008	0.0021	(0.0050,0.1933)
$\theta = 0.1$	150	0.1004	0.0004	0.0016	(0.0232,0.1776)
	200	0.1004	0.0004	0.0011	(0.0335,0.1672)
	250	0.0993	−0.0003	0.0009	(0.0400,0.1593)
	50	0.2962	−0.0038	0.0075	(0.1220,0.4703)
	100	0.2976	−0.0024	0.0039	(0.1744,0.4207)
$\theta = 0.3$	150	0.3014	0.0014	0.0026	(0.2008,0.4020)
	200	0.3012	0.0012	0.0021	(0.2141,0.3884)
	250	0.2999	−0.0001	0.0015	(0.2220,0.3778)
	50	0.4937	−0.0063	0.0062	(0.3335,0.6539)
	100	0.4964	−0.0036	0.0031	(0.3834,0.6094)
$\theta = 0.5$	150	0.4984	−0.0016	0.0023	(0.4063,0.5905)
	200	0.4991	−0.0009	0.0017	(0.4193,0.5788)
	250	0.5006	0.0006	0.0014	(0.4294,0.5718)
	50	0.7012	0.0012	0.0034	(0.5844,0.8180)
	100	0.6987	−0.0013	0.0016	(0.6156,0.7817)
$\theta = 0.7$	150	0.7008	0.0008	0.0012	(0.6333,0.7683)
	200	0.7003	0.0003	0.0009	(0.6418,0.7588)
	250	0.6999	−0.0001	0.0007	(0.6476,0.7523)
	50	0.8972	−0.0027	0.0008	(0.8442,0.9527)
	100	0.8984	−0.0016	0.0004	(0.8601,0.9368)
$\theta = 0.9$	150	0.9010	0.0010	0.0002	(0.8703,0.9318)
	200	0.9001	0.0001	0.0002	(0.8733,0.9269)
	250	0.8998	−0.0002	0.0001	(0.8758,0.9238)

The ρ-inversion estimator, $\hat{\theta}_\rho$, can be deduced by solving equation

$$\hat{\theta}_\rho = h^{-1}(\hat{\rho}),$$

where the increasing function h can be derived from Proposition 6,

$$h(\theta) = \frac{12(1-\theta)^3 \ln(1-\theta) - 3\theta^4 + 22\theta^3 - 30\theta^2 + 12\theta}{\theta^4}$$

and $\hat{\rho}$ denotes the sample Spearman's rho expressed, in terms of $(U_1, V_1), \ldots, (U_n, V_n)$, as follows:

$$\hat{\rho} = \frac{12}{n} \sum_{i=1}^{n} U_i V_i - 3.$$

We let $\mathbf{w} = (u, v)$ and recall basic definitions of the joint and marginal empirical distribution functions,

$$F_n(\mathbf{w}) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{I}_{\{U \le u, V \le v\}}, \; F_{n1}(u) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{I}_{\{U \le u\}}, \; F_{n2}(v) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{I}_{\{V \le v\}}.$$

Following Deheuvels [26], we define the empirical copula by

$$C_n(\mathbf{w}) = F_n(F_{n1}^{-1}(u), F_{n2}^{-1}(v)), \text{ for } (u, v) \in [0,1]^2,$$

where $F_{ni}^{-1}(y) = \inf\{x : F_{ni}(x) \ge y\}$ for $i = 1, 2$.

We define now the k^{th} copula moment $M_k(\theta)$ as the expectation of $(C(\mathbf{W}))^k$, i.e.,

$$M_k(\theta) := E[(C(\mathbf{W}))^k] = \int_{[0,1]^2} (C(\mathbf{w}))^k dC(\mathbf{w}), \quad k = 1, 2, \ldots$$

It is conspicuous that case $k = 1$ corresponds to $M_1(\theta) = (\tau_\theta + 1)/4$. The copula moment estimator adapted to our case, $\hat{\theta}_{\text{CM}}$, is obtained by solving equation

$$M_1(\theta) = \frac{1}{n} \sum_{i=1}^{n} C_n(F_{n1}(u), F_{n2}(v)).$$

For more details on the consistency and asymptotic normality of estimators $\hat{\theta}_\rho$ and $\hat{\theta}_{\text{CM}}$, we refer to [27–29] and the references therein.

To compare the performance of the three estimates mentioned earlier, a simulation study was carried out for some combinations of parameter θ and sample size n. The selection of the true values for parameter θ should be meaningful, ensuring that each value corresponds to a level of dependence: weak, moderate, or strong. If we regard Spearman's ρ as a measure of dependence, we should choose a copula parameter value that aligns with specific ρ values by means of Proposition 6,

$$\rho_\theta = \frac{12(1-\theta)^3 \ln(1-\theta) - 3\theta^4 + 22\theta^3 - 30\theta^2 + 12\theta}{\theta^4}.$$

For each considered value of θ, we generated 500 samples from the underlying copula and computed three estimates, $\hat{\theta}_{\text{ML}}$, $\hat{\theta}_\rho$, and $\hat{\theta}_{\text{CM}}$. Furthermore, the simulation procedure was repeated for different sample sizes n with $n = 50, 100, 150, 200, 250$.

Table 4 summarizes the results of the Monte Carlo simulations by showing the values of the estimated bias and MSE. The maximum likelihood estimator performed better than the other two estimators; it had the smallest bias and the smallest MSE. For medium and strong dependance, it is worth mentioning that MSE($\hat{\theta}_{\text{CM}}$) was smaller than MSE($\hat{\theta}_\rho$). All estimators became more stable since their estimated bias and MSE became smaller as the size of the sample increased.

Table 4. Bias and MSE of ML, CM, and $\rho-$ inversion estimators of θ.

		$\rho = 0.1$		$\rho = 0.5$		$\rho = 0.9$	
		$\theta = 0.1576$		$\theta = 0.6401$		$\theta = 0.9456$	
		Bias	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**
$n = 50$	$\hat{\theta}_{\text{ML}}$	−0.0083	0.0059	−0.0033	0.0041	0.0006	0.0003
	$\hat{\theta}_\rho$	0.0187	0.0237	−0.0100	0.0146	−0.0043	0.0013
	$\hat{\theta}_{\text{CM}}$	0.1067	0.0376	0.0378	0.0104	0.0123	0.0008
$n = 100$	$\hat{\theta}_{\text{ML}}$	−0.0036	0.0030	−0.0024	0.0021	−0.0005	0.0002
	$\hat{\theta}_\rho$	−0.0069	0.0145	−0.0059	0.0075	−0.0016	0.0006
	$\hat{\theta}_{\text{CM}}$	0.0423	0.0165	0.0210	0.0052	0.0061	0.0004
$n = 150$	$\hat{\theta}_{\text{ML}}$	0.0012	0.0020	0.0017	0.0014	−0.0004	0.0001
	$\hat{\theta}_\rho$	−0.0041	0.0117	−0.0012	0.0049	−0.0012	0.0004
	$\hat{\theta}_{\text{CM}}$	0.0284	0.0118	0.0156	0.0036	0.0042	0.0002
$n = 200$	$\hat{\theta}_{\text{ML}}$	−0.0011	0.0014	−0.0013	0.0011	0.0004	0.0001
	$\hat{\theta}_\rho$	−0.0028	0.0089	−0.0013	0.0035	−0.0006	0.0003
	$\hat{\theta}_{\text{CM}}$	0.0235	0.0096	0.0106	0.0026	0.0034	0.0002
$n = 250$	$\hat{\theta}_{\text{ML}}$	−0.0001	0.0012	0.0013	0.0008	−0.0003	0.0001
	$\hat{\theta}_\rho$	0.0014	0.0077	−0.0010	0.0028	−0.0004	0.0003
	$\hat{\theta}_{\text{CM}}$	0.0121	0.0075	0.0083	0.0022	0.0023	0.0001

4. Conclusions

We introduced an innovative method for constructing copulas following well-defined conditions outlined in Assumption 1. Additionally, we explored the upper limit of the generalized family defined in (1), analyzing its properties such as singularity, monotonicity, tail dependence, and association measures. Through a comparative analysis, we showed that the maximum likelihood estimator of dependence parameter θ outperforms both the ρ-inversion and copula moment estimators. Investigating the lower bound of the generalized family remains a direction for future research. Expanding function ϕ in (1) beyond product fg could enhance flexibility and broaden the scope of analysis. This avenue will be addressed in an upcoming paper.

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